

INFINITE VOLUME LIMITS OF POLYMER CHAINS WITH PERIODIC CHARGES

FRANCESCO CARAVENNA, GIAMBATTISTA GIACOMIN, AND LORENZO ZAMBOTTI

ABSTRACT. The aim of this paper is twofold:

- To give an elementary and self-contained proof of an explicit formula for the free energy for a general class of polymer chains interacting with an environment through periodic potentials. This generalizes a result in [6] in which the formula is derived by using the Donsker-Varadhan Large Deviations theory for Markov chains. We exploit instead tools from renewal theory.
- To identify the infinite volume limits of the system. In particular, in the different regimes we encounter transient, null recurrent and positive recurrent processes (which correspond to *delocalized*, *critical* and *localized* behaviors of the trajectories). This is done by exploiting the sharp estimates on the partition function of the system obtained by the renewal theory approach.

The precise characterization of the infinite volume limits of the system exposes a non-uniqueness problem. We will however explain in detail how this (at first) surprising phenomenon is instead due to the presence of a first-order phase transition.

2000 *Mathematics Subject Classification*: 60K35, 82B41, 82B44

Keywords: Random Walks, Markov Renewal Theory, Polymers, Infinite volume limits, Gibbs measures, Phase transitions.

1. INTRODUCTION AND MAIN RESULTS

The real systems that we want to model are schematized in Fig. 1. A linear *polymer*, that is a chain made up of *almost* repetitive units (the *monomers*), fluctuates in a medium constituted by two solvents, A and B, separated by an interface. We say *almost* repetitive because the monomers differ for one property, that we call *charge*, that determines the affinity of the monomer for one or the other solvent (in the figure the charge is considered simply as positive, i.e. A-favorable, or negative, i.e. B-favorable, but in general it may have an intensity which also varies from monomer to monomer).

Let us consider the following two possible scenarios:

- 1– Imagine that there are as many monomers preferring the solvent A as the ones preferring B and that the charges are distributed along the chain in such a way that, roughly, the charges alternate. Then the only configurations with all monomers in their preferred solvent are configurations that stick closely to the interface. This is true even if the matching of charges and solvents is only approximate. If this is what happens, we say that the polymer is localized at the interface.
- 2– The limit of the argument above is that it takes into account only of energetic effects (the charge dependent interaction monomer–solvent). In particular perfect matchings are essentially impossible in large systems with non zero temperature, but imperfect matchings leave open the possibility of observing the localization

phenomenon outlined above. In reality however, if for example the A-favorable monomers outnumber the B-favorable ones, say in a ratio two to one, then it is still true that the polymer may end up optimizing the energetic gain via (possibly imperfect) matchings, but it may also take the different strategy of lying above the interface almost all the time, performing in this way only very imperfect matchings, two over three, but gaining (presumably) very much in fluctuation freedom (the so called *entropic gain*).

The situation is therefore rather unclear and it appears that a non trivial energy-entropy competition is governing the system.

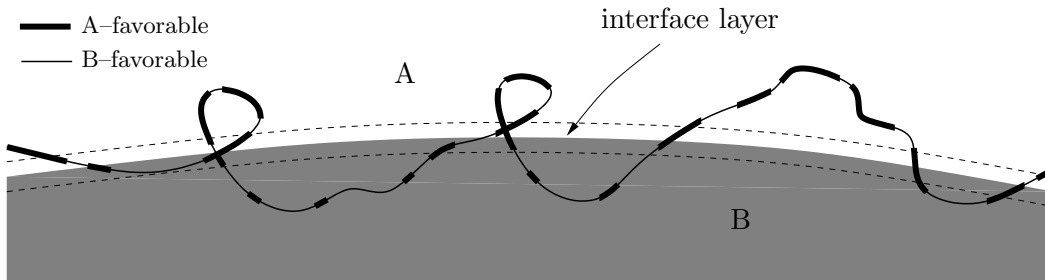


FIGURE 1. A polymer is made up of two types of monomers, type A that lies preferably in the solvent A and type B that prefers solvent B (a thicker line denotes the stretches of A-favorable monomers). In order to satisfy the preferences of all the monomers, the polymer has to keep close to the interface between the two solvents. However entropy plays a role too and observing perfect monomer-solvent matchings is highly improbable as long as the temperature of the system is non zero. A realistic model may also include a different type of interaction, attractive or repulsive, at the interface layer.

The type of polymer we have introduced is what is called a *copolymer*, a synonymous with heterogeneous polymer, and the physical system goes under the name of *copolymer near an interface between selective solvents* [5, 17, 13]. More realistic would be however to consider that the interface is typically non extremely sharp and there is a layer in which the two solvents mix (we could also imagine that in this layer are trapped some impurities) and the monomers may pay a price or receive a reward in crossing, or even lying on, the interface layer. This extra interaction is normally referred to as a *pinning* or *depinning* interaction (but also as *adsorption/desorption*) [24].

One can also imagine the extreme case in which there is no monomer-solvent interaction, but there are only (de)pinning interactions: this is a very realistic situation too, even beyond the two solvent picture we have given. One can in fact imagine that a polymer fluctuates freely in space (or in a solvent), except when it is in proximity of a *defect line* with which it interacts, see e.g. [1] and references therein.

The literature on the realistic situations that we have just outlined is vast. A considerable part of it focuses on a case which is very relevant for applications: the one of periodic distribution of charges (we mention of course also the other extremely relevant case of disordered charge distributions [1, 5, 13]). With this we mean that the sequence of charges repeats after a finite number of monomer units and the polymer is effectively made up piecing together identical stretches of monomers. In this paper we focus exactly on modeling periodic copolymer models with adsorption (or pinning) interactions.

A further important remark is that a polymer model should include the so called *excluded volume interaction*, which leads to self-avoiding walks. We will enforce the self-avoiding condition by making the rather drastic choice of considering directed polymers (of course the trajectory in Fig. 1 may be self-avoiding, if the space is three dimensional).

1.1. The model. We consider a random walk $S := \{S_n\}_{n \in \mathbb{N} \cup \{0\}}$ that is $S_0 = 0$ and $S_n := \sum_{j=1}^n X_j$, where $\{X_j\}_{j \in \mathbb{N}}$ is an IID sequence with $\mathbf{P}(X_1 = +1) = \mathbf{P}(X_1 = -1) = p \in (0, 1/2)$ and $\mathbf{P}(X_1 = 0) = 1 - 2p$ (we have decided to exclude $p = 1/2$ only for notational convenience, because of the periodicity of the walk).

The walk S is our free model. We suppose that S interacts with an environment, that we model with four periodic sequences $\omega^{(+1)}$, $\omega^{(-1)}$, $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$. We consider two models, respectively *free* and *constrained*, defined by

$$\frac{d\mathbf{P}_{N,\omega}^f}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}_N(S))}{\tilde{Z}_{N,\omega}^f} \quad \text{and} \quad \frac{d\mathbf{P}_{N,\omega}^c}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}_N(S))}{\tilde{Z}_{N,\omega}^c} \mathbf{1}_{\{S_N=0\}}, \quad (1.1)$$

where the Hamiltonian is

$$\mathcal{H}_N(S) := \sum_{i=\pm 1} \sum_{n=1}^N \omega_n^{(i)} \mathbf{1}_{\{\text{sign}(S_n)=i\}} + \sum_{n=1}^N \omega_n^{(0)} \mathbf{1}_{\{S_n=0\}} + \sum_{n=1}^N \tilde{\omega}_n^{(0)} \mathbf{1}_{\{\text{sign}(S_n)=0\}}. \quad (1.2)$$

Some comments are in order:

- (1) $\omega^{(\pm 1)}$, $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$ are periodic sequences of real numbers, describing the interaction of the monomers with the solvents and the interface. We say that the sequence $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ is periodic if there exists $T \in \mathbb{N}$ such that $\omega_{n+T} = \omega_n$ for every n . The smallest such T is the period of ω . From now on ω rather denotes the four periodic sequences appearing in (1.2), and we will use $T = T(\omega)$ to denote the least common multiple of the periods of $\omega^{(\pm 1)}$, $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$.
- (2) To define $\text{sign}(S_n)$ when $S_n = 0$ we adopt the following convention: if $S_{n-1} \neq 0$ we set $\text{sign}(S_n) := \text{sign}(S_{n-1})$ while if also $S_{n-1} = 0$ we set $\text{sign}(S_n) := 0$. This definition has the following simple interpretation: $\text{sign}(S_n) = +1, -1$ or 0 according to whether the bond (S_{n-1}, S_n) lies above, below or on the x -axis.
- (3) $\tilde{Z}_{N,\omega}^a := \mathbf{E}[\exp(\mathcal{H}_N) (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}})]$ is the normalization constant, that is usually called *partition function*.
- (4) The measure $\mathbf{P}_{N,\omega}^a$ is invariant under the joint transformation $S \rightarrow -S$ and $\omega^{(+1)} \rightarrow \omega^{(-1)}$, hence we may (and will) assume that

$$h_\omega := \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} (\omega_n^{(+1)} - \omega_n^{(-1)}) \geq 0. \quad (1.3)$$

Remark 1.1 (*Copolymers and pinning models*). The general model (1.1) that we consider will be referred to as a *copolymer with adsorption* model. This includes as special cases the *copolymer* and *pinning* models that were mentioned informally above. More precisely, the copolymer model corresponds to the choice $\omega^{(0)} = \tilde{\omega}^{(0)} \equiv 0$ (this formulation generalizes the case considered in [6]). If instead we set $\omega^{(\pm 1)} \equiv 0$ we are left with the interactions at the interface, or defect line, and we are dealing with a pinning model. We stress that much of the literature on periodic models, e.g. [17, 18, 19] for the copolymer case and [11] for the pinning case, focuses on the case of $T = 2$. We mention as exceptions [21] that deals with the free energy of very particular types of periodic sequences and [22, 23] treating,

in a qualitative and non rigorous fashion, arbitrary T models (see [6] for more details on the literature).

1.2. The free energy and the localization/delocalization alternative. Getting back to the general model (1.1), we observe that from a technical viewpoint it is convenient to set

$$\mathcal{H}'_N(S) := \mathcal{H}_N(S) - \sum_{n=1}^N \omega_n^{(+1)}, \quad (1.4)$$

which just corresponds to $\omega_n^{(+1)} \rightarrow 0$, $\omega_n^{(-1)} \rightarrow (\omega_n^{(-1)} - \omega_n^{(+1)})$, $\tilde{\omega}_n^{(0)} \rightarrow (\tilde{\omega}_n^{(0)} - \omega_n^{(+1)})$, and to note that this new energy yields *the same polymer measures*, namely

$$\frac{d\mathbf{P}_{N,\omega}^a}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}'_N(S))}{Z_{N,\omega}^a} (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}), \quad (1.5)$$

where the new partition function is just $Z_{N,\omega}^a = \tilde{Z}_{N,\omega}^a \exp(-\sum_{n=1}^N \omega_n^{(+1)})$.

It is not difficult to see that $\{\log Z_{N,\omega}^c\}_n$ is a super-additive sequence and from this to establish the existence of the limit

$$F_\omega := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,\omega}^c. \quad (1.6)$$

F_ω is the *free energy* of the system. It is also rather straightforward to show that (1.6) holds also if we replace the superscript c with f , i.e. the free energy does not depend on the boundary condition. For a proof of these facts the reader is referred for example to [13], but we stress that in this paper we will give a proof of the existence of the limit in (1.6) that does not rely on super-additivity, see Section 2.

Leaving aside for the moment the problem of determining F_ω , we focus instead on a simple but crucial aspect of the free energy, namely that

$$F_\omega \geq 0. \quad (1.7)$$

The proof of this fact is absolutely elementary:

$$\begin{aligned} \frac{1}{N} \log Z_{N,\omega}^c &\geq \frac{1}{N} \mathbf{E} [\exp(\mathcal{H}'_N(S)); S_n > 0 \text{ for } n = 1, \dots, N-1] \\ &= \frac{1}{N} \log \left(\frac{1}{2} \exp(\omega_N^{(0)}) K(N) \right) \xrightarrow{N \rightarrow \infty} 0, \end{aligned} \quad (1.8)$$

where we have introduced the notation $K(N) := \mathbf{P}(S_n \neq 0 \text{ for } n = 1, \dots, N-1 \text{ and } S_N = 0)$ and we have used the polynomial decay of $K(\cdot)$. Later on we will need the precise asymptotic behavior of $K(\cdot)$. This can be found in [10, Ch. XII] and we anticipate it here:

$$K(N) \stackrel{N \rightarrow \infty}{\sim} \frac{c_K}{N^{3/2}}, \quad (1.9)$$

for a positive constant c_K that depends on p (by $a_N \sim b_N$ we mean of course $a_N/b_N \rightarrow 1$ as $N \rightarrow \infty$).

Inspired by (1.7) and by its proof, it is customary to say that the system is

- localized if $F_\omega > 0$;
- delocalized if $F_\omega = 0$.

As unsatisfactory as this definition may look at first, we will see in the next paragraph that the above *dichotomy* captures some of the essential features of the system. For the moment we would like to stress that the free energy F_ω admits an *explicit formula* in terms

of the charges ω , see Theorem 2.1 below, that has been first derived in [6], by means of large deviations techniques, and then re-obtained in [8], using a more direct approach based on renewal theory. One of the purposes of this work is to present (in Section 2) a direct self-contained proof of this formula, using renewal theory ideas in analogy to [8]. As we shall see next, the renewal theory approach allows to go much further.

1.3. From free energy to path behavior. A very natural question is whether the localization (resp. delocalization) defined in terms of the free energy does correspond to a real localized (resp. delocalized) behavior of the trajectories of the polymer measure $\mathbf{P}_{N,\omega}^a$. A positive answer to this question had been already given before, but only in terms of weak (de)localization results and leaving out essentially in all instances the critical behavior (see [6] and references therein). We have instead given strong path results in terms of scaling limits in [8], by exploiting renewal theory ideas. Here we pursue the line and obtain the precise characterization of the infinite volume limit of the system.

The key technical point is that we can go well beyond the Laplace asymptotic behavior captured by the free energy. In fact in [8] we have shown that there exists a basic parameter δ_ω , which is an explicit function of the charges ω , that determines the *precise asymptotic behavior* of the partition function (we define δ_ω in (2.7), but the precise expression of δ_ω is not essential now). Let us denote by \mathbb{S} the Abelian group $\mathbb{Z}/(T\mathbb{Z})$, that is $\{0, \dots, T-1\}$ with sum modulo T , and we write equivalently $[n] = \alpha$ or $n \in \alpha$ to denote that n is in the equivalence class of $\alpha \in \mathbb{S}$. The result proven in [8] is:

Theorem 1.2 (Sharp asymptotic estimates). *Fix $\eta \in \mathbb{S}$ and consider the asymptotic behavior of $Z_{N,\omega}^c$ as $N \rightarrow \infty$ along $[N] = \eta$. Then:*

- (1) *If $\delta_\omega < 1$ then $Z_{N,\omega}^c \sim C_{\omega,\eta}^</math> / $N^{3/2}$;$*
- (2) *If $\delta_\omega = 1$ then $Z_{N,\omega}^c \sim C_{\omega,\eta}^=$ / $N^{1/2}$;*
- (3) *If $\delta_\omega > 1$ then $F_\omega > 0$ and $Z_{N,\omega}^c \sim C_{\omega,\eta}^> \exp(F_\omega N)$,*

where the positive quantities F_ω , $C_{\omega,\eta}^>$, $C_{\omega,\eta}^<$ and $C_{\omega,\eta}^=$ are given explicitly in Theorem 3.1.

In Theorem 3.1 one finds also the asymptotic behavior for the free endpoint case. We remark that Theorem 1.2 implies that the *localized regime* corresponds to $\delta_\omega > 1$. The complementary delocalized regime $\delta_\omega \leq 1$ clearly splits in two sub-regimes that we call *strictly delocalized regime* ($\delta_\omega < 1$) and *critical regime* ($\delta_\omega = 1$). The reason for such a denomination is clear if one considers that $\omega \mapsto \delta_\omega$ is a continuous function on the set $\{\omega : T(\omega) = T\}$ (that is for fixed period) and hence arbitrarily small variations in ω may change $\delta_\omega = 1$ to $\delta_\omega > 1$ or $\delta_\omega < 1$, while of course the localized and strictly delocalized regimes are *stable*.

Theorem 1.2 has been applied in [8] to determine the scaling limits of our models. More precisely, it has been shown that for every fixed $\eta \in \mathbb{S}$ the linear interpolation of $\{S_{i/N}/\sqrt{N}\}_{i=0,\dots,N}$ under $\mathbf{P}_{N,\omega}^a$ converges in distribution as $N \rightarrow \infty$ along the subsequence $[N] = \eta$. The properties of the limit process (that in general may depend on the choice of η) are radically different in the three regimes $\delta_\omega \leq 1$ and this gives a precise picture of localization/delocalization (see [8, Th. 1.3]).

It is natural to look at the scaling limits as describing the *global properties* of the system. In this paper we focus rather on the *infinite volume limit* of our model, that is on the weak convergence of the polymer measures $\mathbf{P}_{N,\omega}^a$ *without rescaling*, as a measure on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$. The latter space being equipped with the product topology, weak convergence

simply means convergence of all finite dimensional marginal distributions and hence the infinite volume limit contains the information on the *local properties* of the model.

In the following theorem, that is our main result, we characterize the possible limits of $\mathbf{P}_{N,\omega}^a$, showing that they exhibit distinctive features of localization/delocalization according to whether $\delta_\omega > 1$ or $\delta_\omega < 1$ (the critical case $\delta_\omega = 1$ is borderline, as for the scaling limits).

Theorem 1.3 (Infinite volume limit). *For every $\eta \in \mathbb{S}$ and for $a = f, c$ the polymer measure $\mathbf{P}_{N,\omega}^a$ converges weakly as $N \rightarrow \infty$ along the subsequences $[N] = \eta$ to a limit measure $\mathbf{P}_\omega^{\eta,a}$, law of an irreducible Markov process on \mathbb{Z} which is:*

- *positive recurrent if $\delta_\omega > 1$ (localized regime);*
- *null recurrent if $\delta_\omega = 1$ (critical regime);*
- *transient if $\delta_\omega < 1$ (strictly delocalized regime).*

When $\delta_\omega \geq 1$ the limit law $\mathbf{P}_\omega^{\eta,a} = \mathbf{P}_\omega$ does not depend on η and a , hence both the polymer measures $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge weakly as $N \rightarrow \infty$ to the same limit \mathbf{P}_ω .

We prove this theorem in Section 5, exploiting the precise asymptotic behavior of $Z_{N,\omega}^a$ given in Theorem 1.2 and in Proposition 3.1, and we also provide an explicit construction of the limit law $\mathbf{P}_\omega^{\eta,a}$ in all regimes. We also point out that the transition kernel of the Markov law $\mathbf{P}_\omega^{\eta,a}$ is only *periodically inhomogeneous*, that is $\mathbf{P}_\omega^{\eta,a}(S_{n+1} = y | S_n = x)$ is a T -periodic function of n .

Results similar to Theorems 1.2 and 1.3 have been obtained for homogeneous pinning systems (see [7, 9, 16]) and for periodic pinning models in the $T = 2$ case [17] (we stress however that a $T = 2$ periodic pinning model based on simple random walk becomes, by considering the marginal on odd or even sites, a homogeneous model based on a random walk with jumps in $\{-1, 0, 1\}$: this *decimation procedure* is less straightforward for $T > 2$ and it leads to rather involved models).

In spite of recent advances, see [14, 15] and references therein, obtaining results like Theorem 1.2 and Theorem 1.3 for disordered models appears to be a real challenge (the problem is more apparent for the delocalized regime, but also the localized regime of disordered systems is still only partly understood).

1.4. Non-uniqueness and first order transition. It should have possibly struck the reader the dependence of the infinite volume limit on the *boundary conditions* $[N]$ and on $a = f, c$ in the strictly delocalized regime $\delta_\omega < 1$ (recall that our system is one dimensional!). This *trouble* was already present in [8, Th. 1.3], i.e. for the scaling limits, where however also the critical regime is affected.

Here we are going to clarify this point. First of all we point out that for a large number of cases, that we characterize explicitly in §5.2, all limit laws $\mathbf{P}_\omega^{\eta,a}$ appearing in Theorem 1.3 in fact coincide also in the strictly delocalized regime, hence there is only one infinite volume measure which is the limit of both $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ as $N \rightarrow \infty$. This is true in particular for copolymer and pinning models (defined in Remark 1.1).

However there do exist cases when the laws $\mathbf{P}_\omega^{\eta,a}$ have a real dependence on the boundary conditions $a = f, c$ and $[N] = \eta$ (we anticipate that this happens only for $h_\omega = 0$). In Section 6 we study in detail this phenomenon, showing that all possible limit laws $\mathbf{P}_\omega^{\eta,a}$ are in fact superpositions of *two extremal Gibbs measures* \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- , that we define explicitly and which differ sharply for the asymptotic behavior as $N \rightarrow \infty$: with probability 1, $S_N \rightarrow +\infty$ under \mathbf{Q}_ω^+ and $S_N \rightarrow -\infty$ under \mathbf{Q}_ω^- (we recall that for $\delta_\omega < 1$ the infinite volume process is transient). We insist however on the fact that, in general,

Q_ω^\pm differ also for the statistics of the finitely many returns close to the origin and they are not related by a simple symmetry.

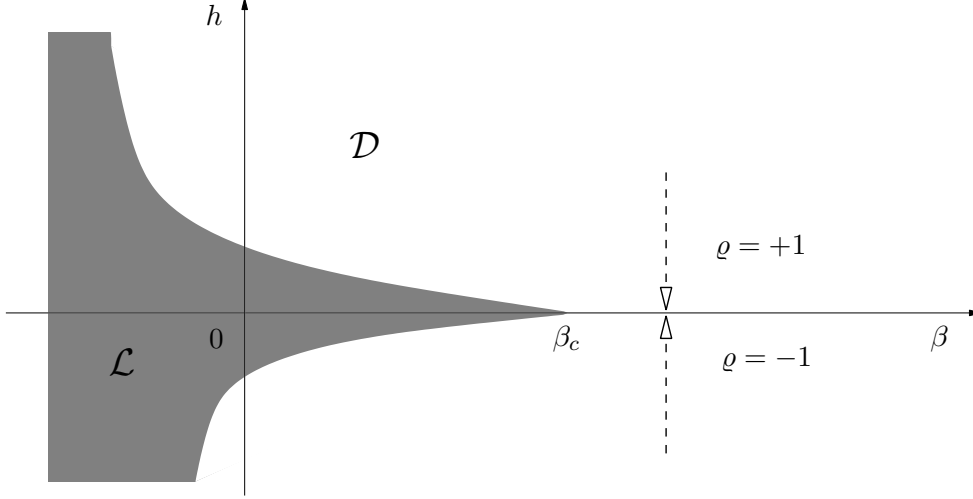


FIGURE 2. A sketch of the phase diagram for the model (1.10). Approaching $h = 0$ in the sense of the dashed arrowed lines, one observes the two sharply different behaviors of paths completely delocalized above ($\varrho = +1$) or below ($\varrho = -1$) the interface. Taking the (weak) limits as $h \searrow 0$ (respectively $h \nearrow 0$) of the infinite volume measures one obtains precisely the measure Q_ω^+ (respectively Q_ω^-). The infinite volume limit for $h = 0$ instead exists only along subsequences since there are in general T different limit points (for the constrained endpoint case and T for the free endpoint case) that are different superpositions of Q_ω^+ and Q_ω^- .

We stress that this multiplicity of infinite volume measures should not be regarded as a pathology, but it is rather the sign of the presence of a *first order phase transition* in the system. In order to be more precise let us consider for instance the case of

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) = \frac{1}{\tilde{Z}_{N,\omega}} \exp \left(\sum_{n=1}^N (\omega_n + h) \text{sign}(S_n) - \beta \sum_{n=1}^N \mathbf{1}_{\{S_n=0\}} \right), \quad (1.10)$$

with h and β two real parameters and ω a fixed centered ($\sum_{n=1}^T \omega_n = 0$) periodic configuration of charges which is non trivial, that is $\omega_i \neq 0$ for some i . For the sake of this paragraph we define the free energy directly by $f(\beta, h) := \lim_{N \rightarrow \infty} N^{-1} \log \tilde{Z}_{N,\omega}$, that is we do not make the transformation (1.4). Then with arguments analogous to (1.8) one gets $f(\beta, h) \geq |h|$ and hence we say that the system is localized if $f(\beta, h) > |h|$ and delocalized if $f(\beta, h) = |h|$.

The phase diagram of such a model is sketched in Figure 2. In particular it is easy to show that for β sufficiently large and positive the system is delocalized for any value of h . On the other hand, for $(\beta, h) = (0, 0)$ the system is localized, see [8, App. B] or [6]. By the monotonicity of the free energy in β , one immediately infers that there exists $\beta_c > 0$ such that at $h = 0$ localization (resp. delocalization) prevails for $\beta < \beta_c$ (resp. for $\beta \geq \beta_c$).

The interesting point is that the delocalized regime that appears when $\beta \geq \beta_c$ has sharply different properties according to the sign of h : in fact, since $f(\beta, h) = |h|$, the

quantity $\varrho(\beta, h) := \partial f(\beta, h)/\partial h$ takes the value $+1$ for $h > 0$ and -1 for $h < 0$. Notice that this quantity has the following direct interpretation ($h \neq 0$):

$$\varrho(\beta, h) := \lim_{N \rightarrow \infty} \mathbf{E}_{N, \omega} \left[\frac{1}{N} \sum_{n=1}^N \text{sign}(S_n) \right]. \quad (1.11)$$

Since for $\beta \geq \beta_c$ the free energy is not differentiable at $h = 0$, the system is said to undergo a *first order phase transition*.

It is worth remarking that first order phase transitions are usually associated to multiple infinite volume limits (*phase coexistence*) like the ones obtained in Theorem 1.3. In our case we are able to assert with precision that \mathbf{Q}_ω^\pm are *pure phases* (that is extremal Gibbs states) and which linear combination of \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- one obtains taking the limits along the subsequences with fixed values of $[N]$.

1.5. Outline of the paper. In Section 2 we give the formula for F_ω and its proof based on renewal theory: along the proof the fundamental processes characterizing the rest of the paper will appear naturally. Section 2 contains only algebraic manipulations and basic probability facts. In Section 3 we recall and discuss a more general version of Theorem 1.2, proven in [8]. In Section 4 we make a number of manipulations on the finite volume polymer measures that clarify the role of the random set of contacts of the polymer with the interface and the excursions of the polymer in the solvents. In Section 5 we identify the infinite volume limits of the system, proving in particular Theorem 1.3. Finally, in Section 6 we unravel the non-uniqueness phenomenon encountered by taking infinite volume limits.

2. A RENEWAL THEORY PATH TO AN EXPLICIT EXPRESSION FOR THE FREE ENERGY

We are going to explain how renewal theory ideas lead to a representation formula for the partition function $Z_{N, \omega}^c$ that we exploit to establish an explicit formula for the free energy F_ω . It will be clear that one can go much beyond with such a formula and we will explain (without a full proof) how to obtain Theorem 1.2 from it.

2.1. The matrix encoding procedure. In order to give the formula for the free energy we need to recall the *matrix encoding procedure* presented in [6]. We recall the definition $\mathbb{S} := \mathbb{Z}/T\mathbb{Z}$ and, for $n \in \mathbb{Z}$, we denote by $[n] \in \mathbb{S}$ the equivalence class of n , that is if $m \in [n]$ there exists $j \in \mathbb{Z}$ such that $m = n + jT$.

The basic structure underlying S is for us the renewal process $\tau := \{\tau_j\}_{j=0,1,\dots}$ defined by $\tau_0 := 0$ and

$$\tau_{j+1} := \inf \{n > \tau_j : S_n = 0\}, \quad (2.1)$$

and, since S is recurrent, $\tau_j < \infty$ for every j , \mathbf{P} -a.s.. The sequence τ , which we will view also as a random subset of $\mathbb{N} \cup \{0\}$, is a renewal process precisely because $\{\tau_j - \tau_{j-1}\}_{j \in \mathbb{N}}$ is an IID sequence. It is therefore fully characterized by the law of τ_1 and we have already set the notation $\mathbf{P}(\tau_1 = n) = K(n)$. Note that, by (1.9), S is only *null* recurrent, since $\mathbf{E}[\tau] = +\infty$. In renewal terms, τ is persistent (but we will prefer to refer to it as recurrent) and in fact *null* persistent.

Next we can define a $\mathbb{S} \times \mathbb{S}$ matrix $\Sigma_{\alpha, \beta}$ by the relation

$$\sum_{n=n_1+1}^{n_2} (\omega_n^{(-1)} - \omega_n^{(+1)}) = -(n_2 - n_1) h_\omega + \Sigma_{[n_1], [n_2]}, \quad (2.2)$$

where h_ω has been defined in (1.3) (we stress that the matrix $\Sigma_{\alpha, \beta}$ is well-defined because the charges ω are T -periodic). In this way we have decomposed the above sum into a drift

term and a fluctuating term, where the latter has the key property of depending on n_1 and on n_2 only through the respective equivalence classes $[n_1]$ and $[n_2]$. Now for $\alpha, \beta \in \mathbb{S}$ and $\ell \in \mathbb{N}$ we define

$$\Phi_{\alpha, \beta}^{\omega}(\ell) := \begin{cases} \omega_{\beta}^{(0)} + (\tilde{\omega}_{\beta}^{(0)} - \omega_{\beta}^{(+1)}) & \text{if } \ell = 1, \ell \in \beta - \alpha \\ \omega_{\beta}^{(0)} + \log \left(\frac{1}{2} \left(1 + \exp(-\ell h_{\omega} + \Sigma_{\alpha, \beta}) \right) \right) & \text{if } \ell > 1, \ell \in \beta - \alpha ; \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

and for $n \in \mathbb{N}$ we introduce the $\mathbb{S} \times \mathbb{S}$ matrix $M^{\omega}(n)$ defined by

$$M_{\alpha, \beta}^{\omega}(n) := e^{\Phi_{\alpha, \beta}^{\omega}(n)} K(n) \mathbf{1}_{(n \in \beta - \alpha)}. \quad (2.4)$$

Summing over $n \in \mathbb{N}$ the entries of M^{ω} we obtain a $\mathbb{S} \times \mathbb{S}$ matrix B :

$$B_{\alpha, \beta} := \sum_{n \in \mathbb{N}} M_{\alpha, \beta}^{\omega}(n). \quad (2.5)$$

We finally introduce for $b \geq 0$ the $\mathbb{S} \times \mathbb{S}$ matrix $A^{\omega}(b)$:

$$A_{\alpha, \beta}^{\omega}(b) := \sum_{n \in \mathbb{N}} M_{\alpha, \beta}^{\omega}(n) \exp(-bn). \quad (2.6)$$

Notice that $A^{\omega}(0) = B$. It is important to note that $A^{\omega}(b)$ is a matrix with positive entries and therefore, by the classical Perron-Frobenius Theorem [2], its spectral radius $Z_{\omega}(b)$ is also a positive eigenvalue, with the property that the corresponding left and right eigenvectors may be chosen to have strictly positive components. Moreover $Z_{\omega}(b)$ has also the property of being simple, that is its eigenspace has dimension one, and it is larger than the absolute value of any other (possibly complex) eigenvalue of $A^{\omega}(b)$.

We know also that $Z_{\omega}(b)$ is a smooth function of b , since $A_{\alpha, \beta}^{\omega}(\cdot)$ is smooth for every α and β , and that $Z_{\omega}(\cdot)$ is also strictly decreasing, since the entries $A_{\alpha, \beta}^{\omega}(\cdot)$ are. The inverse function of $Z_{\omega}(\cdot)$, which is defined on the domain $(0, \delta_{\omega}]$, will be denoted by $Z_{\omega}^{-1}(\cdot)$.

We now introduce the basic positive parameter δ_{ω} , which is defined by

$$\delta_{\omega} := Z_{\omega}(0). \quad (2.7)$$

2.2. A matrix representation and the formula for the free energy. We are now ready to give the explicit formula for the free energy F_{ω} :

Theorem 2.1. *The limit in (1.6) exists and is given by*

$$F_{\omega} = \begin{cases} Z_{\omega}^{-1}(1) & \text{if } \delta_{\omega} > 1 \\ 0 & \text{if } \delta_{\omega} \leq 1 \end{cases}. \quad (2.8)$$

As a preliminary step for the proof of Theorem 2.1 we will make a manipulation on the formula for $Z_{N, \omega}^c$ leading to a particularly useful matrix expression. This is in reality very simple, just set $\iota_N := \sup\{j : \tau_j \leq N\}$ and notice that $\{N \in \tau\} = \{\tau_{\iota_N} = N\}$ is just the event that $\tau_k = N$ for some k . Then, by conditioning on the return times τ and

integrating on the up–down symmetry of the excursions of S , we can write

$$\begin{aligned} Z_{N,\omega}^c &= \mathbf{E} \left[\prod_{j=1}^{\iota_N} \exp(\Phi_{[\tau_{j-1}], [\tau_j]}^\omega(\tau_j - \tau_{j-1})); N \in \tau \right] \\ &= \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \cup \{0\} \\ 0 =: t_0 < t_1 < \dots < t_k := N}} \prod_{j=1}^k M_{[t_{j-1}], [t_j]}^\omega(t_j - t_{j-1}). \end{aligned} \quad (2.9)$$

But we can go further with the following algebraic manipulation: let us denote by $\xi = \xi(b) \in (0, \infty)^\mathbb{S}$ the right eigenvector of $A^\omega(b)$ with eigenvalue $Z_\omega(b)$ (the precise normalization is inessential). Then we introduce the *probability kernel*

$$\Gamma_{\alpha,\beta}(n) := \frac{1}{Z_\omega(b)} \exp(-bn) M_{\alpha,\beta}^\omega(n) \frac{\xi_\beta}{\xi_\alpha}. \quad (2.10)$$

Equation (2.9) may then be rewritten as

$$Z_{N,\omega}^c = \exp(bN) \frac{\xi_{[0]}}{\xi_{[N]}} \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \cup \{0\} \\ 0 =: t_0 < t_1 < \dots < t_k := N}} Z_\omega(b)^k \prod_{j=1}^k \Gamma_{[t_{j-1}], [t_j]}(t_j - t_{j-1}). \quad (2.11)$$

Remark 2.2. We have called Γ *probability kernel* because

$$\sum_{n,\beta} \Gamma_{\alpha,\beta}(n) = \frac{1}{Z(b)} \frac{(A(b)\xi)_\alpha}{\xi_\alpha} = 1, \quad (2.12)$$

because by definition ξ is the right Perron Frobenius eigenvector of $A(b)$ (notice that we have dropped the explicit dependence on ω , something that we will frequently do below). Therefore it is possible to interpret Γ as the transition matrix of a Markov chain on $\mathbb{S} \times \mathbb{N}$ that we denote $\{(J_k, T_k)\}_{k=0,1,\dots}$:

$$\mathbb{P}_b((J_{k+1}, T_{k+1}) = (\beta, n) | (J_k, T_k) = (\alpha, m)) = \Gamma_{\alpha,\beta}(n). \quad (2.13)$$

Note that since this transition probability does not depend on m , this chain may be built by first sampling the $\{J_k\}_k$ process, that is a finite state space (\mathbb{S}) Markov chain with transition matrix $\sum_n \Gamma_{\alpha,\beta}(n)$, and then sampling $\{T_k\}_k$ as independent random variables with distributions $\Gamma_{J_{k-1}, J_k}(\cdot) / \sum_n \Gamma_{J_{k-1}, J_k}(n)$.

Thanks to Remark 2.2 we interpret (2.11) in probabilistic terms: for $J_0 := [0]$ (and $T_0 := 0$ for definiteness, but the value of T_0 is irrelevant) we define the *Markov renewal process* $\hat{\tau}$ as the partial sum process of the sequence $\{T_k\}_k$, that is

$$\hat{\tau}_j := T_1 + \dots + T_j, \quad j \in \mathbb{N}, \quad \hat{\tau}_0 := 0. \quad (2.14)$$

This is a particular case of the general class of Markov renewal processes treated for example in [2]. In terms on this new process, (2.11) takes a nice probabilistic expression:

Lemma 2.3. *For every b we have*

$$Z_{N,\omega}^c = \exp(bN) \frac{\xi_{[0]}}{\xi_{[N]}} \mathbb{E}_b \left[Z_\omega(b)^{\hat{\iota}_N}; N \in \hat{\tau} \right], \quad (2.15)$$

where $\hat{\iota}_N := \inf\{j : \hat{\tau}_j \leq N\} = \max(\hat{\tau} \cap \{0, 1, \dots, N\})$ and we have exploited the fact that $\hat{\tau}$ may be looked upon as a (random) subset of $\mathbb{N} \cup \{0\}$.

The proof of this lemma follows immediately from (2.11), (2.13) and (2.14). Next we pass to the proof of Theorem 2.1, treating separately the three regimes $\delta_\omega \gtrless 1$.

Proof of Theorem 2.1, case $\delta_\omega > 1$. Since $\delta_\omega = Z_\omega(0) > 1$, the image of $Z_\omega(\cdot)$ contains 1 and we can set $b := Z_\omega^{-1}(1)$, so that $Z_\omega(b) = 1$ and (2.15) becomes

$$Z_{N,\omega}^c = \exp(bN) \frac{\xi_{[0]}}{\xi_{[N]}} \mathbb{P}_b(N \in \hat{\tau}). \quad (2.16)$$

Since ξ is a vector with positive entries, (2.16) implies immediately that the superior limit of $\{(1/N) \log Z_{N,\omega}^c\}_N$ is bounded above by b and it suffices to show that $\mathbb{P}_b(N \in \hat{\tau})$ does not vanish exponentially fast in N to establish that the free energy exists and that it takes the value b . However it is rather intuitive that a much better bound holds, namely that there exists $c > 0$ such that

$$\inf_N \mathbb{P}_b(N \in \hat{\tau}) \geq c, \quad (2.17)$$

because since $b > 0$ the process $\hat{\tau}$ is positive recurrent, that is $\sup_j \mathbb{E}_b[T_j] < \infty$. This is in fact a consequence of the Markov Renewal Theorem [2, Th. VII.4.3], which gives the precise asymptotic behavior of $\mathbb{P}_b(N \in \hat{\tau})$ as $N \rightarrow \infty$. More directly, it suffices to remark that the processes $\hat{\tau}^\beta := \{\hat{\tau}_j : J_j = \beta\}$ is a classical (i.e. no Markov dependence) positive recurrent renewal process and that $\mathbb{P}_b(N \in \hat{\tau}) = \mathbb{P}_b(N \in \hat{\tau}^\beta)$ for $\beta = [N]$. Therefore the classical Renewal Theorem yields $\mathbb{P}_b(N \in \hat{\tau}^\beta) \rightarrow T/\mathbb{E}_b[\hat{\tau}_2^\beta - \hat{\tau}_1^\beta] > 0$ as $N \rightarrow \infty$ along the subsequence $[N] = \beta$, and (2.17) is proven, because there are only finitely many options for β . \square

Proof of Theorem 2.1, case $\delta_\omega = 1$. Since $\delta_\omega = Z_\omega(0) = 1$, also in this case 1 is in the image of $Z_\omega(\cdot)$ and we set $b = Z_\omega^{-1}(1) = 0$. In particular, $\limsup_N (1/N) \log Z_{N,\omega}^c \leq b = 0$ like before, but we cannot proceed like above for a lower bound, since, under \mathbb{P}_0 , $\hat{\tau}$ is null recurrent (that is $\mathbb{E}_0[\hat{\tau}_j - \hat{\tau}_{j-1}] = \infty$). However, by (1.8), we already know that $\liminf_N (1/N) \log Z_{N,\omega}^c \geq 0$ and we are done. \square

Proof of Theorem 2.1, case $\delta_\omega < 1$. This is quick too: since $\delta_\omega = Z_\omega(0) < 1$, by choosing $b = 0$ in (2.15) we clearly see that $Z_{N,\omega}^c = O(1)$, so $\limsup_N (1/N) \log Z_{N,\omega}^c \leq 0$, and (1.8) provides the lower bound. Note that in this case to the Markov renewal process is superimposed a killing rate $Z_\omega(0)$ and it is this *transient* or *terminating* process that we should consider as the Markov renewal process naturally associated to the regime in which $Z_\omega(0) < 1$ (this point will emerge clearly in Section 5). \square

Remark 2.4. The proof we just completed implicitly contains the most fundamental ideas of this work, but also of [8]. Theorem 1.2 should appear now as the natural sharpening of this proof: it is clear that it requires sharp estimates on suitable *mass renewal functions*, that is $\mathbb{P}_b(N \in \hat{\tau})$ and that the three regimes, corresponding to positive recurrent, null recurrent and transient Markov renewals, require different techniques and they in fact yield very different results. We also stress that the Markov renewal processes arising in the three regimes are not mere technical tools: they are in fact the limiting processes given in Section 1.3.

3. SHARP ASYMPTOTIC ESTIMATES

The aim of this section is to report a more detailed version of Theorem 1.2, collecting the results obtained in Section 3 of [8], see Theorem 3.1 below.

We recall from the last section the notation $\xi = \xi(b)$ for the right Perron–Frobenius eigenvector of the matrix $A(b)$, defined in (2.6). More explicitly:

$$\sum_{\gamma} A_{\alpha,\gamma}(b) \xi_{\gamma} = Z(b) \xi_{\alpha}, \quad \forall \alpha \in \mathbb{S}, \quad (3.1)$$

where we recall that $Z(b)$ is the Perron–Frobenius eigenvalue of $A(b)$. We choose ξ in $(0, \infty)^{\mathbb{S}}$ and we fix the normalization $\sum_{\gamma} \xi_{\gamma} = 1$. As in the proof of Theorem 2.1, we observe that when $\delta_{\omega} \geq 1$ the image of $Z(\cdot)$ contains 1 and hence we can set $b := Z^{-1}(1) = F_{\omega}$. From now on, we will always mean that when $\delta_{\omega} \geq 1$ the eigenvector $\xi = \xi(b)$ is evaluated for $b = F_{\omega}$ (when $\delta_{\omega} < 1$ we do not need to use the eigenvector).

Theorem 3.1 (Sharp asymptotic estimates). *Let $k \in \mathbb{N}$ with $[k] = \alpha$. Then as $N \rightarrow \infty$ along $[N] = \eta$ we have:*

- (1) *If $\delta_{\omega} > 1$ then there exist constants $c_{\eta}^a > 0$, $a = f, c$, such that:*

$$Z_{N-k, \theta_k \omega}^a \sim (c_{\eta}^a \xi_{\alpha}) \exp(F_{\omega}(N-k)). \quad (3.2)$$

- (2) *If $\delta_{\omega} = 1$ then there exist constants $\kappa_{\eta}^a > 0$, $a = f, c$, such that:*

$$Z_{N-k, \theta_k \omega}^c \sim (\kappa_{\eta}^c \xi_{\alpha}) \frac{1}{\sqrt{N}}, \quad Z_{N-k, \theta_k \omega}^f \rightarrow (\kappa_{\eta}^f \xi_{\alpha}). \quad (3.3)$$

- (3) *If $\delta_{\omega} < 1$ then there exist constants $\Lambda_{\alpha, \eta}^a > 0$, $a = f, c$, such that:*

$$Z_{N-k, \theta_k \omega}^c \sim \Lambda_{\alpha, \eta}^c \frac{1}{N^{3/2}}, \quad Z_{N-k, \theta_k \omega}^f \sim \Lambda_{\alpha, \eta}^f \frac{1}{N^{1/2}}. \quad (3.4)$$

The precise value of the constants $\{c_{\eta}^a, \kappa_{\eta}^a, \Lambda_{\alpha, \eta}^a\}$ is given in [8, §3.2, §3.3, §3.4] and that of $\Lambda_{\alpha, \eta}^a$ also in (5.8) below. Here we notice that for $\delta_{\omega} \geq 1$ the prefactor in the asymptotic behavior of $Z_{N-k, \theta_k \omega}^a$ is equal to a constant, depending on η and a , multiplied by the eigenvector ξ_{α} : this fact will be important in the proof of Proposition 5.2 below. On the other hand, for $\delta_{\omega} < 1$ in general the constant $\Lambda_{\alpha, \eta}^a$ does not admit such a factorization and this is the source of the dependence of the infinite volume limit on the boundary conditions $a = f, c$ and $[N] = \eta$. This phenomenon, anticipated in §1.4, is studied in detail in Section 6.

4. THE POLYMER MEASURE: CONTACT SET AND EXCURSIONS

In this section we perform a preliminary analysis of the polymer measure $\mathbf{P}_{N, \omega}^a$ that will be a basic tool for the proof of Theorem 1.3, given in the next section.

The starting point is a very useful decomposition of $\mathbf{P}_{N, \omega}^a$. The intuitive idea is that a path $\{S_n\}_{n \leq N}$ can be split into two main ingredients:

- the family $\{\tau_k\}_{k=0,1,\dots}$ of *returns to zero* of S , already introduced in (2.1);
- the family of *excursions from zero* $\{S_{i+\tau_{k-1}} : 0 \leq i \leq \tau_k - \tau_{k-1}\}_{k=1,2,\dots}$

Moreover, since each excursion can be either positive or negative, it is also useful to consider separately the signs of the excursions $\sigma_k := \text{sign}(S_{\tau_{k-1}+1})$ and the absolute values $\{e_k(i) := |S_{i+\tau_{k-1}}| : i = 1, \dots, \tau_k - \tau_{k-1}\}$. Observe that these are trivial for an excursion with length 1: in fact if $\tau_k = \tau_{k-1} + 1$ then $\sigma_k = 0$ and $e_k(0) = e_k(1) = 0$.

Remark 4.1. A word about definiteness: if $\tau_k = +\infty$ (and hence $\tau_i = +\infty$ for all $i \geq k$), the definition of the variables σ_i and $e_i(\cdot)$ given above do not make sense for $i > k$. However the problem is immaterial, since in this case these variables are irrelevant for the purpose of reconstructing the path $\{S_n\}_n$, and consequently we agree to define σ_i and $e_i(\cdot)$ for $i > k$ in an arbitrary way.

The process $(\tau_k)_k$ can be also viewed as a (random) subset of $\mathbb{N} \cup \{0\}$, and for this reason we will refer to it as to the *contact set* (of course we have in mind the polymer interpretation of our model described in the introduction). The crucial point, already exploited in [8] to obtain the scaling limits of our model, is the following description of the law of the contact set and of the excursions under the polymer measure $\mathbf{P}_{N,\omega}^a$.

4.1. The contact set. We recall the definition $\iota_N = \sup\{k : \tau_k \leq N\}$. Let us first consider the returns $(\tau_k)_k$ under $\mathbf{P}_{N,\omega}^a$. The law of this process can be viewed as a probability measure $\mathbf{p}_{N,\omega}^a$ on the class \mathcal{A}_N of subsets of $\{1, \dots, N\}$: indeed for $A \in \mathcal{A}_N$, writing

$$A = \{t_1, \dots, t_{|A|}\}, \quad 0 =: t_0 < t_1 < \dots < t_{|A|} \leq N, \quad (4.1)$$

we can set

$$\mathbf{p}_{N,\omega}^a(A) := \mathbf{P}_{N,\omega}^a(\tau_i = t_i, i \leq \iota_N). \quad (4.2)$$

The measure $\mathbf{p}_{N,\omega}^a$ describes the set of contacts of the polymer with the interface. From the inclusion of \mathcal{A}_N into $\{0, 1\}^{\mathbb{N} \cup \{0\}}$, the family of all subsets of $\mathbb{N} \cup \{0\}$, $\mathbf{p}_{N,\omega}^a$ can be viewed as a measure on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$ (this observation will be useful in the following).

Let us describe more explicitly $\mathbf{p}_{N,\omega}^a(A)$, using the (strong) Markov property of $\mathbf{P}_{N,\omega}^a$. We use throughout the paper the notation (4.1). Recalling the definition (2.4) of $M_{\alpha,\beta}(t)$, we have for $a = c, f$:

$$\begin{aligned} \mathbf{p}_{N,\omega}^a(\{k_0, \dots, k_n\}) &= \mathbf{P}_{N,\omega}^a(\tau_1 = k_1, \dots, \tau_n = k_n) \\ &= \left[\prod_{i=1}^n M_{[k_{i-1}], [k_i]}(k_i - k_{i-1}) \right] \frac{Z_{N-k_n, \theta_{k_n} \omega}^a}{Z_{N,\omega}^a}, \end{aligned} \quad (4.3)$$

for all $0 =: k_0 < k_1 < \dots < k_n \leq N$ and $a = c, f$.

4.2. The signs. From the very definition (1.5) of our model it is easy to check that, conditionally on $\{\iota_N, (\tau_j)_{j \leq \iota_N}\}$, the signs $(\sigma_k)_{k \leq \iota_N}$ are under $\mathbf{P}_{N,\omega}^a$ an independent family. For $k \leq \iota_N$, the conditional law of σ_k is specified by:

- if $\tau_k = 1 + \tau_{k-1}$, then $\sigma_k = 0$;
- if $\tau_k > 1 + \tau_{k-1}$, then σ_k can take the two values ± 1 with

$$\mathbf{P}_{N,\omega}^a(\sigma_k = +1 \mid (\tau_j)_{j \leq \iota_N}) = \frac{1}{1 + \exp\{-(\tau_k - \tau_{k-1})h_\omega + \Sigma_{[\tau_{k-1}], [\tau_k]}\}}. \quad (4.4)$$

Observe that when $\tau_{\iota_N} < N$ (which can happen only for $a = f$) there is a last (incomplete) excursion in the interval $\{0, \dots, N\}$, and the sign of this excursion is also expressed by (4.4) for $k = \iota_N + 1$, provided we set $\tau_{\iota_N+1} := N$.

4.3. The moduli of the excursions. Again, from the definition of our model it follows that, conditionally on $\{\iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1}\}$, the excursions $(e_k(\cdot))_{k \leq \iota_N+1}$ are under $\mathbf{P}_{N,\omega}^a$ an independent family. For $k \leq \iota_N$, the conditional law of $e_k(\cdot)$ on the event $\{\tau_{k-1} = \ell_0, \tau_k = \ell_1\}$ is specified for $f = (f_i)_{i=0,\dots,\ell_1-\ell_0}$ by

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a \left(e_k(\cdot) = f \mid \iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1} \right) \\ &= \mathbf{P} \left(S_i = f_i : i = 0, \dots, \ell_1 - \ell_0 \mid S_i > 0 : i = 1, \dots, \ell_1 - \ell_0 - 1, S_{\ell_1 - \ell_0} = 0 \right). \end{aligned} \quad (4.5)$$

For $a = f$, when $\tau_{\iota_N} < N$ the conditional law on the event $\{\tau_{\iota_N} = \ell < N\}$ of the last incomplete excursion $e_{\iota_N+1}(\cdot)$ is specified for $f = (f_i)_{i=0,\dots,N-\ell}$ by

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a \left(e_{\iota_N+1}(\cdot) = f \mid \iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1} \right) \\ &= \mathbf{P} \left(S_i = f_i : i = 0, \dots, N - \ell \mid S_i > 0 : i = 1, \dots, N - \ell \right). \end{aligned} \quad (4.6)$$

4.4. Building the infinite volume measure. We stress that the above descriptions of the contact set, of the signs and of the moduli of the excursions fully characterize the polymer measure $\mathbf{P}_{N,\omega}^a$. A remarkable fact is that, conditionally on $(\tau_k)_{k \geq 0}$, the joint distribution of $(\sigma_j, e_j)_{j \leq \iota_N}$ does not depend on N : in this sense, the N -dependence is contained in the contact set law $\mathbf{p}_{N,\omega}^a$.

For this reason, the next section is devoted to the study of the asymptotic behavior of the contact set measure $\mathbf{p}_{N,\omega}^a$ as $N \rightarrow \infty$. The main result is that, for every $\eta \in \mathbb{S}$, the measure $\mathbf{p}_{N,\omega}^a$ converges weakly on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$, as $N \rightarrow \infty$ along the subsequence $[N] = \eta$, toward a limit measure $\mathbf{p}_\omega^{\eta,a}$ (which in general depends on a and η).

From this result and from the above considerations, one would like to infer that the full polymer measure $\mathbf{P}_{N,\omega}^a$ converges weakly on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$, as $N \rightarrow \infty$ along $[N] = \eta$, toward a limit measure $\mathbf{P}_\omega^{\eta,a}$ which is constructed by *pasting the excursion over the limit contact set*. This is indeed true when the cardinality of the contact set $\{\tau_n\}_n$ is infinite under the limit contact set law $\mathbf{p}_\omega := \mathbf{p}_\omega^{\eta,a}$, that is when $\mathbf{p}_\omega(\tau_k < +\infty) = 1$ for all $k \geq 0$ (we will see that this is what happens when $\delta_\omega \geq 1$). In this case the infinite volume polymer measure $\mathbf{P}_\omega := \mathbf{P}_\omega^{\eta,a}$ can be completely reconstructed from \mathbf{p}_ω (to lighten the notation, for the rest of this section the dependence of $\mathbf{P}_\omega^{\eta,a}$ and $\mathbf{p}_\omega^{\eta,a}$ on a and η will be omitted).

However when $\delta_\omega < 1$ it turns out that the cardinality of the contact set is \mathbf{p}_ω -a.s. *finite*, hence there is a last infinite excursion. In this case to obtain the weak convergence of the full polymer measure $\mathbf{P}_{N,\omega}^a$ it is also necessary to determine the law of the sign of the last infinite excursion. But let us describe more in detail how to construct the infinite volume polymer measure \mathbf{P}_ω .

The proper case. We consider first the case when $\mathbf{p}_\omega(\tau_k < +\infty) = 1$ for all $k \geq 0$. Then the infinite volume polymer measure \mathbf{P}_ω is the law on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$ under which the processes $(\tau_j)_j$, $(\sigma_j)_j$ and $(e_j(\cdot))_j$ have the following laws:

- The process $(\tau_j)_j$ is drawn according to \mathbf{p}_ω .
- Conditionally on $(\tau_j)_j$, the variables $(\sigma_j)_j$ are independent. The conditional law of σ_k depends only on (τ_{k-1}, τ_k) and it is specified in the following way:
 - if $\tau_k - \tau_{k-1} = 1$, then $\sigma_k = 0$;
 - if $\tau_k - \tau_{k-1} > 1$, then σ_k takes the two values ± 1 with probabilities given by the r.h.s. of (4.4).

- Conditionally on $(\tau_j, \sigma_j)_j$, the variables $(e_j(\cdot))_j$ are independent. The conditional law of $e_k(\cdot)$ on the event $\{\tau_{k-1} = \ell_0, \tau_k = \ell_1\}$ is given by the r.h.s. of (4.5).

Of course these requirements determine uniquely the law \mathbf{P}_ω .

The defective case. Next we analyze the defective case, when the cardinality of the set $\{\tau_n\}_n$ is \mathbf{p}_ω -a.s. finite, which is what happens when $\delta_\omega < 1$.

Let us denote by $\rho := \sup\{k \geq 0 : \tau_k < +\infty\}$ the index of the last point in the contact set, and by assumption we have $\mathbf{p}_\omega(\rho < +\infty) = 1$. In this case to characterize the infinite volume polymer measure \mathbf{P}_ω it suffice to specify the laws of the processes $(\tau_j)_{j \in \mathbb{N} \cup \{0\}}$, $(\sigma_j)_{j=1, \dots, \rho+1}$ and $(e_j(\cdot))_{j=1, \dots, \rho+1}$ under \mathbf{P}_ω .

As before, the process $(\tau_j)_j$ is drawn according to the law \mathbf{p}_ω . Conditionally on $(\tau_j)_j$ the variables $(\sigma_j)_{j=1, \dots, \rho+1}$ are independent, and conditionally on $(\tau_j, \sigma_j)_j$, the variables $(e_j(\cdot))_{j=1, \dots, \rho+1}$ are independent: therefore it remains to specify the conditional laws of σ_k and of $e_k(\cdot)$, for $k = 1, \dots, \rho + 1$. However it is easy to see that for $k \leq \rho$ there is still no change with respect to the proper case, that is the conditional laws are given by the r.h.s. of (4.4) and (4.5) respectively. Hence we are left with specifying the conditional laws of the last sign $\sigma_{\rho+1}$ and of the last modulus $e_{\rho+1}(\cdot)$.

For the last modulus the answer is rather intuitive: on the event $\tau_{\rho+1} = \ell$, the conditional law of $e_{\rho+1}(\cdot)$ is given for any $n \in \mathbb{N}$ and for $f = (f_i)_{i=0, \dots, n}$ by:

$$\begin{aligned} \mathbf{P}_\omega \left(e_k(i) = f_i : i = 0, \dots, n \mid (\tau_j, \sigma_j)_j \right) &= \mathbf{P}^\uparrow \left(S_i = f_i : i = 0, \dots, n \right) \\ &:= \lim_{N \rightarrow \infty} \mathbf{P} \left(S_i = f_i : i = 0, \dots, n \mid S_i > 0 : i = 1, \dots, N \right), \end{aligned} \quad (4.7)$$

where the existence of such limit is well known, cf. [3].

On the other hand, the law of the sign of the last excursion $\sigma_{\rho+1}$ has to be determined by a direct computation and this will be done in §5.2. Once this is done, the construction of the measure \mathbf{P}_ω in the defective case is complete. A remarkable fact is that, for the choice of free or constrained boundary conditions, the law of $\sigma_{\rho+1}$ is in fact determined by \mathbf{p}_ω . However this is not true in general: one can show (we will not pursue this point in detail) that more general boundary conditions may yield different infinite volume measures, having the same law for the contact set but a different law for the sign of the last infinite excursion.

5. INFINITE VOLUME LIMITS

This section contains the proof of Theorem 1.3. We study the limit as $N \rightarrow \infty$ of the polymer measure $\mathbf{P}_{N, \omega}^a$, using the sharp asymptotic behavior of the partition function given in Theorem 3.1. We recall that $\mathbf{P}_{N, \omega}^a$ is a probability measure on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$ and that we endow the latter space with the product topology, hence weak convergence means convergence of all finite dimensional marginal distributions.

Our focus is mainly on the contact set law $\mathbf{p}_{N, \omega}^a$, defined in (4.2), which is a measure on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$. We are going to show that, for $a = f, c$, for any fixed $\eta \in \mathbb{S}$ and for any value of δ_ω , the measure $\mathbf{p}_{N, \omega}^a$ converges weakly on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$ as $N \rightarrow \infty$ along the subsequence $[N] = \eta$. When $\delta_\omega \geq 1$ the convergence actually holds true without having to impose the $[N] = \eta$ constraint, while when $\delta_\omega < 1$ the limit may really depend on the value of η and of $a = f, c$ (in §5.2 we characterize precisely the instances in which this happens).

Once the convergence of $\mathbf{p}_{N, \omega}^a$ (as $N \rightarrow \infty$ along $[N] = \eta$) is proven, the analogous statement for the polymer measure $\mathbf{P}_{N, \omega}^a$ follows by the arguments given in §4.4.

Remark 5.1. In the proof we actually show that, under the limit measure of $\mathbf{p}_{N,\omega}^a$, the process $\{\tau_k\}_{k \geq 0}$ is a *Markov renewal process* with *modulating chain* $\{J_k\}_{k \geq 0} := \{\lceil \tau_k \rceil\}_{k \geq 0}$. This means that, setting $T_k := \tau_k - \tau_{k-1}$ for $k \in \mathbb{N}$, the joint process $\{(J_k, T_k)\}_{k \in \mathbb{N}}$ is a Markov chain on $\mathbb{S} \times \mathbb{N}$ such that the transition probability to go from (J_k, T_k) to (J_{k+1}, T_{k+1}) does not depend on J_k :

$$\mathbb{P}((J_{k+1}, T_{k+1}) = (\beta, n) | (J_k, T_k) = (\alpha, m)) = \Gamma_{\alpha, \beta}(n), \quad (5.1)$$

see also Remark 2.2 and the lines that follow it. The transition kernel $\Gamma_{\alpha, \beta}(n)$ is called the *semi-Markov kernel* of the Markov renewal process $\{\tau_k\}$. We are going to find an explicit expression for $\Gamma_{\alpha, \beta}(n)$, showing in particular that the laws of the T_k are:

- (1) integrable if $\delta_\omega > 1$ (localized regime);
- (2) defective if $\delta_\omega < 1$ (strictly delocalized regime);
- (3) non integrable if $\delta_\omega = 1$ (critical regime).

A detailed account on Markov renewal processes can be found in [2].

Next we pass to the proof of Theorem 1.3. For ease of exposition, we consider first the cases $\delta_\omega > 1$ and $\delta_\omega = 1$, where there are no problems of uniqueness, and then the more delicate strictly delocalized regime $\delta_\omega < 1$.

5.1. The regimes ($\delta_\omega > 1$) and ($\delta_\omega = 1$). We are going to prove the following:

Proposition 5.2. *If $\delta_\omega \geq 1$ then the polymer measures $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge as $N \rightarrow \infty$ to the same limit \mathbf{P}_ω , under which $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel $(\Gamma_{\alpha, \beta}(x) : \alpha, \beta \in \mathbb{S}, x \in \mathbb{N})$, defined by:*

$$\Gamma_{\alpha, \beta}(x) := M_{\alpha, \beta}(x) e^{-F_\omega x} \frac{\xi_\beta}{\xi_\alpha}. \quad (5.2)$$

We recall that $F_\omega = 0$ if $\delta_\omega \leq 1$ and $F_\omega > 0$ if $\delta_\omega > 1$.

Proof of Proposition 5.2. By the asymptotic behavior of $Z_{N,\omega}^a$ in (3.2) and (3.3) above, we have for all $\alpha, \eta \in \mathbb{S}$ and $k \in \alpha$:

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \frac{Z_{N-k, \theta_k \omega}^a}{Z_{N,\omega}^a} = e^{-F_\omega k} \frac{\xi_{[k]}}{\xi_{[0]}}, \quad (5.3)$$

and since the right hand side does not depend on η , then the limit exists as $N \rightarrow \infty$.

By (4.3) it follows that for $0 =: k_0 < k_1 < \dots < k_j$, $a = c, f$:

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbf{p}_{N,\omega}^a(\{k_0, \dots, k_j\}) &= \left[\prod_{i=1}^j M_{[k_{i-1}], [k_i]}(k_i - k_{i-1}) \right] e^{-F_\omega k_j} \frac{\xi_{[k_j]}}{\xi_{[0]}} \\ &= \prod_{i=1}^j \Gamma_{[k_{i-1}], [k_i]}(k_i - k_{i-1}), \end{aligned} \quad (5.4)$$

and this shows that $\mathbf{p}_{N,\omega}^a$ converges weakly on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$ as $N \rightarrow \infty$ toward the law \mathbf{p}_ω under which $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel $\Gamma_{\alpha, \beta}(x)$.

Notice that $\sum_{x \in \mathbb{N}, \beta \in \mathbb{S}} \Gamma_{\alpha, \beta}(x) = 1$, that is $\mathbf{p}_\omega(\tau_k < +\infty) = 1$ for all $k \geq 0$. Therefore the weak convergence of the full polymer measure $\mathbf{P}_{N,\omega}^a$ on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$ follows from the arguments given in §4.4, and the proof is completed. \square

5.2. **The regime** ($\delta_\omega < 1$). We introduce the subset of ω defined by

$$\mathcal{P}^< := \{\omega : \delta_\omega < 1, \quad h_\omega = 0, \quad \exists \alpha, \beta : \Sigma_{\alpha, \beta} \neq 0\}, \quad (5.5)$$

where h_ω and $\Sigma_{\alpha, \beta}$ have been defined respectively in (1.3) and (2.2). We are going to prove that when $\omega \notin \mathcal{P}^<$ both the free and the constrained polymer measures $\mathbf{P}_{N, \omega}^a$, $a = f, c$, converge weakly as $N \rightarrow \infty$, without having to impose the constraint $[N] = \eta$, while for $\omega \in \mathcal{P}^<$ the limit exists as $N \rightarrow \infty$ along $[N] = \eta$ and in general depend on the choice of a and η . It is worth stressing that for the two motivating models introduced in Remark 1.1, the pinning and the copolymer models, ω never belongs to $\mathcal{P}^<$. This is clear for the pinning case, where by definition $\Sigma \equiv 0$. On the other hand, in the copolymer case it is known that if $h_\omega = 0$ and $\exists \alpha, \beta : \Sigma_{\alpha, \beta} \neq 0$ then $\delta_\omega > 1$, cf. [8, App. B].

It will turn out that in the strictly delocalized regime there exists a.s. a last return to zero, i.e. the process $(\tau_k)_{k \geq 0}$ is defective. In order to express this with the language of Markov renewal processes, we introduce the sets $\bar{\mathbb{S}} := \mathbb{S} \cup \{\infty\}$ and $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, extending the equivalence relation to $\bar{\mathbb{N}}$ by $[\infty] = \infty$.

We need some notation: we set

$$\tilde{L}_{\alpha, \beta} := \begin{cases} c_K (1 + \exp(\Sigma_{\alpha, \beta})) & \text{if } h_\omega = 0 \\ c_K & \text{if } h_\omega > 0 \end{cases}, \quad L_{\alpha, \beta} := \frac{1}{2} \exp(\omega_\beta^{(0)}) \tilde{L}_{\alpha, \beta}. \quad (5.6)$$

We notice that for any ω :

$$L_{\alpha, \beta} := \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} M_{\alpha, \beta}(x). \quad (5.7)$$

In [8, §3.4] it is proven that the constants $\Lambda_{\alpha, \eta}^a$ appearing in (3.4) are equal to:

$$\Lambda_{\alpha, \eta}^c = [(1 - B)^{-1} L (1 - B)^{-1}]_{\alpha, \eta}, \quad \Lambda_{\alpha, \eta}^f = [(1 - B)^{-1} \tilde{L}]_{\alpha, \eta}, \quad (5.8)$$

where B is defined in (2.5). Finally we set for all $\alpha, \eta \in \mathbb{S}$:

$$\mu_{\alpha, \eta}^c := [L (1 - B)^{-1}]_{\alpha, \eta}, \quad \mu_{\alpha, \eta}^f := \tilde{L}_{\alpha, \eta}, \quad (5.9)$$

and for all $\eta \in \mathbb{S}$ and $a = f, c$ we introduce the semi-Markov kernel on $\bar{\mathbb{S}} \times \bar{\mathbb{N}}$:

$$\Gamma_{\alpha, \beta}^{\eta, a}(x) := \begin{cases} M_{\alpha, \beta}(k) \Lambda_{\beta, \eta}^a / \Lambda_{\alpha, \eta}^a & \alpha \in \mathbb{S}, \quad x \in \mathbb{N}, \quad \beta = [x] \in \mathbb{S} \\ \mu_{\alpha, \eta}^a / \Lambda_{\alpha, \eta}^a & \alpha \in \mathbb{S}, \quad x = \infty, \quad \beta = [\infty] \\ 1 & \alpha = \beta = [\infty], \quad x = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (5.10)$$

Notice that $\Gamma^{\eta, a}$ is really a semi-Markov kernel, since for $\alpha \in \mathbb{S}$:

$$\begin{aligned} \sum_{\beta \in \bar{\mathbb{S}}} \sum_{x \in \bar{\mathbb{N}}} \Gamma_{\alpha, \beta}^{\eta, a}(x) &= \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \sum_{\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} \frac{M_{\alpha, \beta}(x) \Lambda_{\beta, \eta}^a}{\Lambda_{\alpha, \eta}^a} = \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \frac{1}{\Lambda_{\alpha, \eta}^a} [B \cdot \Lambda^a]_{\alpha, \eta} \\ &= \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \frac{1}{\Lambda_{\alpha, \eta}^a} (\Lambda_{\alpha, \eta}^a - \mu_{\alpha, \eta}^a) = 1. \end{aligned}$$

We are going to prove the following:

Proposition 5.3. *Let $\delta_\omega < 1$ and $\eta \in \mathbb{S}$. Then:*

- (1) for $a = f, c$, $\mathbf{P}_{N,\omega}^a$ converges weakly as $N \rightarrow \infty$ along $[N] = \eta$ toward a measure $\mathbf{P}_\omega^{\eta,a}$, under which $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel given by $\Gamma_{\alpha,\beta}^{\eta,a}(x)$.
- (2) if $\omega \notin \mathcal{P}^<$, then $\mathbf{P}_\omega^{\eta,a} =: \mathbf{P}_\omega$ and $\Gamma^{\eta,a} =: \Gamma^<$ depend neither on η nor on a , and both $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge as $N \rightarrow \infty$ to \mathbf{P}_ω , under which $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel $\Gamma^<$.

Remark 5.4. Part (2) of Proposition 5.3 is an easy consequence of part (1). In fact from equation (5.6) it follows immediately that when $\omega \notin \mathcal{P}^<$ then both matrices $(L_{\alpha,\beta})$ and $(\tilde{L}_{\alpha,\beta})$ are constant in α , and therefore Λ^a factorizes into a tensor product, i.e.

$$\Lambda_{\alpha,\eta}^a = \lambda_\alpha^a \nu_\eta^a, \quad \alpha, \eta \in \mathbb{S}, \quad (5.11)$$

where $(\lambda_\alpha^a)_{\alpha \in \mathbb{S}}$ and $(\nu_\alpha^a)_{\alpha \in \mathbb{S}}$ are easily computed. But then it is immediate to check that the semi-Markov kernel $\Gamma^{\eta,a} =: \Gamma^<$ depends neither on η nor on a .

Proof of Proposition 5.3. By the preceding Remark it suffices to prove part (1). By (3.4) we have we have for all $\alpha, \eta \in \mathbb{S}$ and $k \in \alpha$:

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \frac{Z_{N-k, \theta_k \omega}^a}{Z_{N,\omega}^a} = \frac{\Lambda_{[k],\eta}^a}{\Lambda_{[0],\eta}^a}. \quad (5.12)$$

By (4.3) it follows that for $0 =: k_0 < k_1 < \dots < k_j < \infty$, $a = c, f$:

$$\begin{aligned} \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{p}_{N,\omega}^a(\{k_0, \dots, k_j\}) &= \left[\prod_{i=1}^j M_{[k_{i-1}], [k_i]}(k_i - k_{i-1}) \right] \frac{\Lambda_{[k_j],\eta}^a}{\Lambda_{[0],\eta}^a} \\ &= \prod_{i=1}^j \Gamma_{[k_{i-1}], [k_i]}^{\eta,a}(k_i - k_{i-1}). \end{aligned} \quad (5.13)$$

This shows that $\mathbf{p}_{N,\omega}^a$ converges weakly on $\{0, 1\}^{\mathbb{N} \cup \{0\}}$ as $N \rightarrow \infty$, $[N] = \eta$, toward the law $\mathbf{p}_\omega^{\eta,a}$ under which $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel $\Gamma_{\alpha,\beta}^{\eta,a}(x)$.

However this time the semi-Markov kernel is defective, that is $\sum_{\beta \in \mathbb{S}, x \in \mathbb{N}} \Gamma_{\alpha,\beta}^{\eta,a}(x) < 1$, hence the contact set $\{\tau_k\}_{k \geq 0}$ is $\mathbf{p}_\omega^{\eta,a}$ -a.s. unbounded. By the arguments given in §4.4, to obtain the weak convergence of the full polymer measure $\mathbf{P}_{N,\omega}^a$, as $N \rightarrow \infty$ along $[N] = \eta$, toward a limit law $\mathbf{P}_\omega^{\eta,a}$, it remains to determine the law of the sign $\sigma_{\rho+1}$ of the last (infinite) excursion (the notation has been introduced in §4.4).

We start with the free case. We want to show that $\mathbf{P}_{N,\omega}^f(S_N > 0)$ has a limit as $N \rightarrow \infty$ along $[N] = \eta$. By conditioning on the last zero before N we get

$$\mathbf{P}_{N,\omega}^f(S_N > 0) = \frac{1}{Z_{N,\omega}^f} \sum_{k \geq 0} \sum_{\gamma \in \mathbb{S}} \sum_{n=0}^{N-1} M_{[0],\gamma}^{k*}(n) \left(\frac{1}{2} \sum_{t > N-n} K(t) \right),$$

where $K(\cdot)$ has been defined before (1.9) and M^{k*} denotes the convolution of the kernel M with itself k times, the convolution between two kernels F and G being defined by

$$(F * G)_{\alpha,\beta}(n) := \sum_{m=1}^{n-1} \sum_{\gamma \in \mathbb{S}} F_{\alpha,\gamma}(m) G_{\gamma,\beta}(n-m).$$

Therefore, using (1.9) and (3.4) and recalling the definition (2.5), we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ [N] = \eta}} \mathbf{P}_{N,\omega}^f(S_N > 0) = \frac{c_K}{\Lambda_{[0],\eta}^f} \sum_{\gamma \in \mathbb{S}} (1 - B^{-1})_{[0],\gamma}. \quad (5.14)$$

Next we consider the constrained case, where we focus instead on $\mathbf{P}_{N,\omega}^c(S_{\lfloor N/2 \rfloor} > 0)$. Conditioning on the last zero before and on the first zero after $\lfloor N/2 \rfloor$, we can write

$$\begin{aligned} & \mathbf{P}_{N,\omega}^c(S_{\lfloor N/2 \rfloor} > 0) \\ &= \frac{1}{Z_{N,\omega}^c} \sum_{k,h \geq 0} \sum_{\gamma, \zeta \in \mathbb{S}} \sum_{n=0}^{\lfloor N/2 \rfloor} \sum_{m=\lceil N/2 \rceil}^{N-1} M_{[0],\gamma}^{k*}(n) \left(\frac{1}{2} K(m-n) e^{\omega_\zeta^{(0)}} \right) M_{\zeta,[N]}^{h*}(N-m), \end{aligned}$$

and using again (1.9) and (3.4) we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ [N] = \eta}} \mathbf{P}_{N,\omega}^c(S_{\lfloor N/2 \rfloor} > 0) = \frac{c_K}{2\Lambda_{[0],\eta}^c} \sum_{\gamma \in \mathbb{S}} (1 - B^{-1})_{[0],\gamma} \sum_{\zeta \in \mathbb{S}} e^{\omega_\zeta^{(0)}} (1 - B^{-1})_{\zeta,\eta}. \quad (5.15)$$

Now it is easy to check that (5.14) and (5.15) give exactly the probability, under the infinite volume polymer measure $\mathbf{P}_\omega^{\eta,a}$, that the sign $\sigma_{\rho+1}$ of the last (infinite) excursion equals +1, and this completes the proof. \square

6. NON UNIQUENESS OF THE INFINITE VOLUME MEASURE

We want to show that all infinite volume measures $\mathbf{P}_\omega^{\eta,a}$ appearing in the strictly delocalized regime $\delta_\omega < 1$, see Theorem 1.3 and Section 5, are in reality superpositions of only two measures \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- , that are *extremal Gibbs measures* for our system. We split the exposition in two parts:

- in §6.1 we show, by purely combinatorial arguments, that the law of the *contact set* under $\mathbf{P}_\omega^{\eta,a}$ is a superposition of two basic laws \mathbf{q}_ω^+ and \mathbf{q}_ω^- ;
- in §6.2 we show that \mathbf{q}_ω^+ and \mathbf{q}_ω^- can be extended to two laws \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- for the whole process $\{S_n\}_n$ which are extremal Gibbs measure for our system.

6.1. Decomposition of the contact set law. Let $\mathbf{p}_\omega^{\eta,a}$ denote the law of the contact set $(\tau_k)_{k \geq 0}$ under the infinite volume measure $\mathbf{P}_\omega^{\eta,a}$. As it has been shown in §5.2, under $\mathbf{p}_\omega^{\eta,a}$ the process $(\tau_k)_{k \geq 0}$ is a Markov renewal process with semi-Markov kernel $\Gamma^{\eta,a}$, defined in (5.10). More explicitly, for every $n \in \mathbb{N}$ and for all $0 =: k_0 < k_1 < \dots < k_n$ we have:

$$\begin{aligned} \mathbf{p}_\omega^{\eta,a}(\{k_1, \dots, k_n\}) &= \Gamma_{[0],[k_1]}^{\eta,a}(k_1) \cdots \Gamma_{[k_{n-1}],[k_n]}^{\eta,a}(k_n - k_{n-1}) \\ &= M_{[0],[k_1]}(k_1) \cdots M_{[k_{n-1}],[k_n]}(k_n - k_{n-1}) \frac{\Lambda_{[k_n],\eta}^a}{\Lambda_{[0],\eta}^a}, \end{aligned} \quad (6.1)$$

where $\Lambda_{\alpha,\beta}^a$ is defined in (5.8) and the basic kernel $M_{\alpha,\beta}(n)$ has been introduced in (2.4).

To express the law $\mathbf{p}_\omega^{\eta,a}$ as a superposition we are going to exploit the algebraic structure of (6.1). However the steps are more transparent if carried out in a general setting, and one is led to the following definition: we introduce the set \mathcal{C} defined by

$$\mathcal{C} := \left\{ v \in (0, \infty)^\mathbb{S} : \sum_{\beta \in \mathbb{S}} \left(\sum_{n \in \mathbb{N}} M_{\alpha,\beta}(n) \right) v_\beta \leq v_\alpha, \quad \forall \alpha \in \mathbb{S} \right\}. \quad (6.2)$$

More explicitly, we recall that $B_{\alpha,\beta} := \sum_{n \in \mathbb{N}} M_{\alpha,\beta}(n)$ has spectral radius $\delta_\omega < 1$, and therefore we have

$$\mathcal{C} = \left\{ v = (I - B)^{-1} w, \ w \in [0, \infty)^{\mathbb{S} \setminus \{0\}} \right\}. \quad (6.3)$$

The reason for such a definition is that if (and only if) $v \in \mathcal{C}$ then the kernel $M_{\alpha,\beta}(n) \cdot v_\beta / v_\alpha$ is a (defective) semi-Markov kernel, that is $\sum_{\beta, n} M_{\alpha,\beta}(n) v_\beta / v_\alpha \leq 1$ for every $\alpha \in \mathbb{S}$. Therefore, for all $v \in \mathcal{C}$, we can define a (defective) law \mathbf{q}^v for the contact set $\{\tau_k\}_{k \in \mathbb{N}}$ by

$$\mathbf{q}^v(\{k_1, \dots, k_n\}) := M_{[0],[k_1]}(k_1) \cdots M_{[k_{n-1}],[k_n]}(k_n - k_{n-1}) \cdot \frac{v_{[k_n]}}{v_{[0]}}, \quad (6.4)$$

for every $n \in \mathbb{N}$ and for all $0 =: k_0 < k_1 < \dots < k_n$.

Now let us take two arbitrary vectors $v^+, v^- \in \mathcal{C}$. Since \mathcal{C} is a convex set, for all $p \in [0, 1]$ the vector $v := pv^+ + (1-p)v^-$ belongs to \mathcal{C} , hence the law \mathbf{q}^v is well-defined. The crucial result is expressed by the following combinatorial lemma.

Lemma 6.1. *The law $\mathbf{q}^{pv^+ + (1-p)v^-}$ is a superposition of the laws \mathbf{q}^{v^+} and \mathbf{q}^{v^-} :*

$$\mathbf{q}^v = r \mathbf{q}^{v^+} + (1-r) \mathbf{q}^{v^-}, \quad \text{where} \quad r = \frac{pv_{[0]}^+}{pv_{[0]}^+ + (1-p)v_{[0]}^-} \in [0, 1]. \quad (6.5)$$

Proof. By (6.4), all we have to verify is that for every $\alpha \in \mathbb{S}$

$$r \frac{v_\alpha^+}{v_{[0]}^+} + (1-r) \frac{v_\alpha^-}{v_{[0]}^-} = \frac{pv_\alpha^+ + (1-p)v_\alpha^-}{pv_{[0]}^+ + (1-p)v_{[0]}^-}. \quad (6.6)$$

By the definition (6.5) of q , we can rewrite the l.h.s. above as

$$\begin{aligned} & \frac{pv_{[0]}^+}{pv_{[0]}^+ + (1-p)v_{[0]}^-} \frac{v_\alpha^+}{v_{[0]}^+} + \frac{(1-p)v_{[0]}^-}{pv_{[0]}^+ + (1-p)v_{[0]}^-} \frac{v_\alpha^-}{v_{[0]}^-} = \\ & = \frac{pv_\alpha^+}{pv_{[0]}^+ + (1-p)v_{[0]}^-} + \frac{(1-p)v_\alpha^-}{pv_{[0]}^+ + (1-p)v_{[0]}^-} = \frac{pv_\alpha^+ + (1-p)v_\alpha^-}{pv_{[0]}^+ + (1-p)v_{[0]}^-}, \end{aligned} \quad (6.7)$$

and the proof is completed. \square

Next we come back to our model. We define two vectors $v^+(\omega)$ and $v^-(\omega)$ by

$$v^+(\omega)_\alpha := \sum_{\gamma \in \mathbb{S}} (1 - B)_{\alpha,\gamma}^{-1} \quad v^-(\omega)_\alpha := \sum_{\gamma \in \mathbb{S}} (1 - B)_{\alpha,\gamma}^{-1} e^{-\Sigma_{[0],\gamma}} \quad (6.8)$$

where $B_{\alpha,\beta} = B_{\alpha,\beta}^\omega := \sum_{n \in \mathbb{N}} M_{\alpha,\beta}^\omega(n)$, see (2.5), and $\Sigma_{\alpha,\beta}$ is defined in (2.2). From (6.3) we have that $v^\pm(\omega) \in \mathcal{C}$, and the corresponding laws $\mathbf{p}^{v^\pm(\omega)}$ will be simply denoted by \mathbf{q}_ω^\pm :

$$\mathbf{q}_\omega^+ := \mathbf{q}^{v^+(\omega)} \quad \mathbf{q}_\omega^- := \mathbf{q}^{v^-(\omega)}. \quad (6.9)$$

We are ready to state the main result of this paragraph.

Proposition 6.2. *For every $a = \text{f, c}$ and $\eta \in \mathbb{S}$, the measures $\mathbf{p}_\omega^{\eta,a}$ are superpositions of the two laws \mathbf{q}_ω^+ and \mathbf{q}_ω^- :*

$$\mathbf{p}_\omega^{\eta,a} = r(\eta, a, \omega) \mathbf{q}_\omega^+ + (1 - r(\eta, a, \omega)) \mathbf{q}_\omega^-, \quad (6.10)$$

with $r(\eta, a, \omega) \in (0, 1)$.

Proof. We introduce the vector $v(\eta, a, \omega)_\alpha := \Lambda_{\alpha, \eta}^a$ (the dependence of $\Lambda_{\alpha, \beta}^a$ on ω has not been explicitly indicated, but of course is present), and notice that the law $\mathbf{p}_\omega^{\eta, a}$ coincides with $\mathbf{q}^{v(\eta, a, \omega)}$, cf. (6.1) and (6.4).

To prove that $\mathbf{p}_\omega^{\eta, a} = \mathbf{q}^{v(\eta, a, \omega)}$ is a superposition of $\mathbf{q}_\omega^\pm = \mathbf{q}^{v^\pm(\omega)}$, we are going to exploit Lemma 6.1. Let us be more precise: we are going to show that, for every $a = \text{f, c}$ and $\eta \in \mathbb{S}$, the vector $v(\eta, a, \omega)$ is a *linear combination* of two vectors $v^+(\omega)$ and $v^-(\omega)$ with positive coefficients:

$$v(\eta, a, \omega) = x v^+(\omega) + y v^-(\omega), \quad x, y \in \mathbb{R}^+. \quad (6.11)$$

Then the vector $v(\eta, a, \omega)$ can be written as the following *convex combination*:

$$v(\eta, a, \omega) = \frac{x}{x+y} w^+(\omega) + \frac{y}{x+y} w^-(\omega), \quad w^\pm(\omega) := (x+y) v^\pm(\omega), \quad (6.12)$$

and Lemma 6.1 yields that $\mathbf{q}^{v(\eta, a, \omega)} = \mathbf{p}_\omega^{\eta, a}$ is a superposition of the two laws $\mathbf{q}^{w^\pm(\omega)}$. However it is straightforward to see from (6.4) that the laws $\mathbf{q}^{w^\pm(\omega)}$ are the same as $\mathbf{q}^{v^\pm(\omega)}$, because the vectors $v^\pm(\omega)$ and $w^\pm(\omega)$ differ only by a scale factor. Therefore from (6.11) it follows indeed that $\mathbf{p}_\omega^{\eta, a}$ is a superposition of \mathbf{q}_ω^\pm , that is what we have to prove.

Therefore it only remains to show that (6.11) holds true, where of course $x = x(\eta, a, \omega)$ and $y = y(\eta, a, \omega)$. We consider first the constrained case $a = \text{c}$: from the definition (5.8) of $\Lambda_{\alpha, \eta}^{\text{c}}$ and from the definition (5.6) of the matrix L , we can write for $\alpha \in \mathbb{S}$

$$\begin{aligned} v(\text{c}, \eta, \omega)_\alpha &= \Lambda_{\alpha, \eta}^{\text{c}} = [(1-B)^{-1} L (1-B)^{-1}]_{\alpha, \eta} \\ &= \frac{c_K}{2} \sum_{\gamma, \zeta \in \mathbb{S}} (1-B)_{\alpha, \gamma}^{-1} (1 + \exp(\Sigma_{\gamma, \zeta})) e^{\omega_\zeta^{(0)}} (1-B)_{\zeta, \eta}^{-1}. \end{aligned}$$

Observing that $\Sigma_{\gamma, \zeta} = \Sigma_{[0], \zeta} - \Sigma_{[0], \gamma}$ and recalling the definition (6.8) of $v^\pm(\omega)$ we obtain

$$v(\text{c}, \eta, \omega)_\alpha = \left(\frac{c_K}{2} \sum_{\zeta \in \mathbb{S}} e^{\omega_\zeta^{(0)}} (1-B)_{\zeta, \eta}^{-1} \right) v^+(\omega)_\alpha + \left(\frac{c_K}{2} \sum_{\zeta \in \mathbb{S}} e^{\omega_\zeta^{(0)} + \Sigma_{[0], \zeta}} (1-B)_{\zeta, \eta}^{-1} \right) v^-(\omega)_\alpha,$$

which shows that (6.11) holds true for $a = \text{c}$ and gives an explicit expression for $x(\text{c}, \eta, \omega)$ and $y(\text{c}, \eta, \omega)$. With analogous (and simpler) arguments, for the free case we get

$$v(\text{f}, \eta, \omega)_\alpha = c_K v^+(\omega)_\alpha + (c_K e^{\Sigma_{[0], \eta}}) v^-(\omega)_\alpha.$$

Thus (6.11) holds true also for $a = \text{f}$, with $x(\text{f}, \eta, \omega) = c_K$ and $y(\text{f}, \eta, \omega) = c_K e^{\Sigma_{[0], \eta}}$, and the proof is completed. \square

Finally, we observe that one can obtain an explicit formula for the weight $r(\eta, a, \omega)$ appearing in (6.10). From the expression for r given in (6.5) and from (6.12) it follows that

$$\begin{aligned} r(\eta, a, \omega) &= \frac{x(\eta, a, \omega) v^+(\omega)_{[0]}}{x(\eta, a, \omega) v^+(\omega)_{[0]} + y(\eta, a, \omega) v^-(\omega)_{[0]}} \\ &= \frac{x(\eta, a, \omega) v^+(\omega)_{[0]}}{v(\eta, a, \omega)_{[0]}} = \frac{x(\eta, a, \omega) v^+(\omega)_{[0]}}{\Lambda_{[0], \eta}^a}, \end{aligned}$$

having used (6.11) and the definition $v(\eta, a, \omega)_\alpha := \Lambda_{\alpha, \eta}^a$. Observe that the precise values of $x(\eta, a, \omega)$ is the coefficient of $v^+(\omega)_\alpha$ in the last two equations of the proof of Proposition 6.2, cf. (6.11). Then, recalling the definition (6.8) of $v^\pm(\omega)$, we obtain the following

formula for $r(\eta, a, \omega)$: for the constrained case $a = c$

$$r(\eta, c, \omega) = \frac{\sum_{\gamma \in \mathbb{S}} (1 - B)_{[0], \gamma}^{-1} \cdot \frac{c_K}{2} \sum_{\zeta \in \mathbb{S}} e^{\omega_{\zeta}^{(0)}} (1 - B)_{\zeta, \eta}^{-1}}{\Lambda_{[0], \eta}^c}, \quad (6.13)$$

and for the free case $a = f$

$$r(\eta, f, \omega) = \frac{\sum_{\gamma \in \mathbb{S}} (1 - B)_{[0], \gamma}^{-1} \cdot c_K}{\Lambda_{[0], \eta}^f}. \quad (6.14)$$

The exact value of $r(\eta, a, \omega)$ will be important in the next paragraph.

6.2. Extremal Gibbs measures. The aim of this paragraph is to show that the decomposition of the contact set law $\mathbf{p}_{\omega}^{\eta, a}$ in terms of the two laws $\mathbf{q}_{\omega}^{\pm}$, proved in the previous paragraph, can be lifted from the space of the contact set $\{\tau_n\}_n$ to the space of trajectories of $\{S_n\}_n$. More precisely, we are going to show that for all $a = f, c$ and $\alpha \in \mathbb{S}$ the infinite volume measure $\mathbf{P}_{\omega}^{\eta, a}$ is a superposition of two laws $\mathbf{Q}_{\omega}^{\pm}$, depending only on ω , which have $\mathbf{q}_{\omega}^{\pm}$ as contact set laws and which are extremal Gibbs measures for our system.

Let us first recall some basic notions. A measure \mathbf{Q} on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$ is said to be a *Gibbs measure* for our system if it satisfies the so-called DLR equation, that in our setting reads as follows: for all $M \in \mathbb{N}$ and for all $A \subset \mathbb{Z}^M$ we have

$$\mathbf{Q}((S_1, \dots, S_M) \in A \mid S_M) = \mathbf{P}_{M, \omega}^f((S_1, \dots, S_M) \in A \mid S_M) \quad \mathbf{Q}\text{-a.s.} \quad (6.15)$$

The set of all Gibbs measures is clearly a convex set, that is if \mathbf{Q}_1 and \mathbf{Q}_2 are Gibbs measure and $p \in [0, 1]$ then the convex combination $p\mathbf{Q}_1 + (1 - p)\mathbf{Q}_2$ is a Gibbs measure too. If a Gibbs measure \mathbf{Q} cannot be written as a nontrivial convex combination of two distinct Gibbs measures, then \mathbf{Q} is said to be *extremal*. The standard reference on Gibbs measures is [12].

Both the free and the constrained polymer measures $\mathbf{P}_{N, \omega}^f$ and $\mathbf{P}_{N, \omega}^c$ satisfy relation (6.15) for any $M \leq N$. Then it is not a surprise that any weak limit of $\mathbf{P}_{N, \omega}^a$, as $N \rightarrow \infty$ along a subsequence, satisfies (6.15) for all $M \in \mathbb{N}$, that is it is a Gibbs measure, cf. [12, Th. 4.17]. In particular, all infinite volume measures $\mathbf{P}_{\omega}^{\eta, a}$ for $a = f, c$ and $\eta \in \mathbb{S}$, that are found in Theorem 1.3, are Gibbs measures.

The basic Gibbs measures $\mathbf{Q}_{\omega}^{\pm}$ extending \mathbf{q}^{\pm} are introduced in the next lemma.

Lemma 6.3. *There exist two extremal Gibbs measures \mathbf{Q}_{ω}^+ and \mathbf{Q}_{ω}^- such that the law of the contact set $(\tau_n)_{n \geq 0}$ under $\mathbf{Q}_{\omega}^{\pm}$ is exactly $\mathbf{q}_{\omega}^{\pm}$. Moreover these laws satisfy*

$$\mathbf{Q}_{\omega}^+ \left(\lim_{N \rightarrow \infty} S_N = +\infty \right) = 1 \quad \mathbf{Q}_{\omega}^- \left(\lim_{N \rightarrow \infty} S_N = -\infty \right) = 1. \quad (6.16)$$

The proof of this lemma is given below. Now that we have introduced the two laws $\mathbf{Q}_{\omega}^{\pm}$, we are ready to state and prove the main result of this section.

Proposition 6.4. *For all $a = f, c$ and $\eta \in \mathbb{S}$, the infinite volume measures $\mathbf{P}_{\omega}^{\eta, a}$ given in Theorem 1.3, for $\delta_{\omega} < 1$, are superpositions of the two laws \mathbf{Q}_{ω}^+ and \mathbf{Q}_{ω}^- given in Lemma 6.3. More precisely:*

$$\mathbf{P}_{\omega}^{\eta, a} = r(\eta, a, \omega) \mathbf{Q}_{\omega}^+ + (1 - r(\eta, a, \omega)) \mathbf{Q}_{\omega}^-, \quad (6.17)$$

where the weight $r(\eta, a, \omega) \in (0, 1)$ is given by (6.13) and (6.14) for $a = c, f$ respectively.

Proof. We already know by Proposition 6.2 that relation (6.17) holds true if restricted to events involving only the contact set, see (6.10). Now notice that, conditionally on the level set, the law of the signs and of the moduli of the excursions (except for the last infinite one) are *the same* under the three laws $\mathbf{P}_\omega^{\eta,a}$, \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- , that is they are given by (4.4) and (4.5): this is just because all three laws are Gibbs measures for our system and hence satisfy the relation (6.15). Therefore relation (6.17) holds true if restricted to the events that happen not later than the last contact point (more precisely, restricted on the σ -field $\sigma(\tau_\rho, S_k : 0 \leq k \leq \tau_\rho)$, where $\rho := \sup\{k \geq 0 : \tau_k < +\infty\}$ is the index of the last contact point).

Then it remains to focus on the sign $\sigma_{\rho+1}$ and on the modulus $e_{\rho+1}(\cdot)$ of the last (infinite) excursion (the notation has been introduced in §4). For the modulus $e_{\rho+1}(\cdot)$ there are no problems, because it has the same law under each of $\mathbf{P}_\omega^{\eta,a}$, \mathbf{Q}_ω^+ and \mathbf{Q}_ω^- , see (4.7). About the sign $\sigma_{\rho+1}$, we know from Lemma 6.3 that under \mathbf{Q}_ω^+ it is $+1$ and under \mathbf{Q}_ω^- it is -1 , hence under the r.h.s. of (6.17) the variable $\sigma_{\rho+1}$ takes the values $+1$ and -1 with probabilities respectively equal to $r(\eta, a, \omega)$ and $1 - r(\eta, a, \omega)$. However, the l.h.s. of (6.17), that is $\mathbf{P}_\omega^{\eta,a}$, gives exactly the same law to $\sigma_{\rho+1}$, cf. (5.14) and (5.15) with (6.14) and (6.13), and this completes the proof. \square

Proof of Lemma 6.3. Let us introduce two modified finite volume polymer measures $\mathbf{P}_{N,\omega}^+$ and $\mathbf{P}_{N,\omega}^-$, defined by

$$\frac{d\mathbf{P}_{N,\omega}^+}{d\mathbf{P}}(S) := \frac{\exp(\mathcal{H}'_N(S))}{Z_{N,\omega}^+} \mathbf{1}_{(S_N > 0)} \quad \frac{d\mathbf{P}_{N,\omega}^-}{d\mathbf{P}}(S) := \frac{\exp(\mathcal{H}'_N(S))}{Z_{N,\omega}^-} \mathbf{1}_{(S_N < 0)}, \quad (6.18)$$

and notice that $Z_{N,\omega}^\pm = Z_{N,\omega}^f \cdot \mathbf{P}_{N,\omega}^f(S_N \geq 0)$, cf. (1.5). Then from Theorem 3.1 and equation (5.14) it follows that for any fixed $k \geq 0$, as $N \rightarrow \infty$ along $[N] = \eta$

$$Z_{N-k,\theta_k\omega}^+ \sim \left(\sum_{\gamma \in \mathbb{S}} (1 - B)_{[k],\gamma}^{-1} \right) \frac{c_K}{\sqrt{N}} \quad Z_{N-k,\theta_k\omega}^- \sim \left(\sum_{\gamma \in \mathbb{S}} (1 - B)_{[k],\gamma}^{-1} e^{-\Sigma_{[0],\gamma}} \right) \frac{c_K e^{\Sigma_{[0],\eta}}}{\sqrt{N}}.$$

Therefore for every fixed $k \geq 0$ we obtain

$$\exists \lim_{N \rightarrow \infty} \frac{Z_{N-k,\theta_k\omega}^+}{Z_{N,\omega}^+} = \frac{v^+(\omega)_{[k]}}{v^+(\omega)_{[0]}} \quad \exists \lim_{N \rightarrow \infty} \frac{Z_{N-k,\theta_k\omega}^-}{Z_{N,\omega}^-} = \frac{v^-(\omega)_{[k]}}{v^-(\omega)_{[0]}},$$

where the vectors $v^\pm(\omega)$ have been defined in (6.8). But then, following closely the proof of Proposition 5.3, it is easy to prove that both the measures $\mathbf{P}_{N,\omega}^\pm$ converge weakly on $\mathbb{Z}^{\mathbb{N} \cup \{0\}}$ as $N \rightarrow \infty$ toward two limit measures, that we denote by \mathbf{Q}_ω^\pm , such that the contact set $\{\tau_n\}_{n \geq 0}$ under \mathbf{Q}_ω^\pm has law \mathbf{q}_ω^\pm , cf. (6.9). In particular, the cardinality of the contact set $\{\tau_n\}_{n \geq 0}$ is \mathbf{Q}_ω^\pm -a.s. finite. Moreover, by the definition (6.18) of $\mathbf{P}_{N,\omega}^\pm$, it follows that the sign of the last (infinite) excursion under \mathbf{Q}_ω^+ (resp. under \mathbf{Q}_ω^-) is deterministic and takes the value $+1$ (resp. -1). This proves (6.16).

Being weak limit of finite volume polymer measures with suitable boundary conditions, the two laws \mathbf{Q}^\pm are automatically Gibbs measures for our system, cf. [12, Th. 4.17]. To complete the proof, it only remains to show that they are extremal, and by [12, Th. 7.7] this is equivalent to showing that they are trivial on the tail σ -field of the sequence $\{S_n\}_{n \geq 0}$.

Let us denote by $\mathcal{G}_n := \sigma(S_k : k \geq n)$ the σ -field generated by the variables $\{S_k\}$ with index $k \geq n$. We recall that the tail σ -field \mathcal{T} is defined by $\mathcal{T} := \bigcap_{m \in \mathbb{N}} \mathcal{G}_m$. Let us denote

by Θ^{-1} the *inverse shift* defined on \mathcal{G}_1 , that is for $A \in \mathcal{G}_1$ the event $\Theta A \in \mathcal{G}_0$ is defined by

$$(S_0, S_1, S_2, \dots) \in \Theta^{-1}A \iff (S_1, S_2, S_3, \dots) \in A.$$

By iteration we can define the n -shift Θ^{-n} on \mathcal{G}_n , in particular if $A \in \mathcal{T}$ then $\Theta^{-n}A$ is well defined for all $n \in \mathbb{N}$ and $\Theta^{-n}A \in \mathcal{T}$.

We have to show that $\mathbf{Q}_\omega^\pm(A) = 0$ for all $A \in \mathcal{T}$, and for conciseness we focus on \mathbf{Q}_ω^+ (the case \mathbf{Q}_ω^- is analogous). We recall that $\rho := \sup\{k \geq 0 : \tau_k < +\infty\}$ denotes the index of the last contact point, and we stress that $\mathbf{Q}_\omega^+(\rho < +\infty) = 1$. We also recall from §4.4 that the last excursion $\{e_{\rho+1}(k)\}_{k \geq 0} := \{S_{\tau_\rho+k}\}_{k \geq 0}$ has under \mathbf{Q}_ω^+ the law \mathbf{P}^\uparrow of the *random walk conditioned to stay positive*, see (4.7) and [3]. We point out that \mathbf{P}^\uparrow is the law of a Markov chain on $\mathbb{N} \cup \{0\}$ which is transient: $\mathbf{P}^\uparrow(\lim_{N \rightarrow \infty} S_N = +\infty) = 1$, cf. [3].

By conditioning on the value of the last contact point, we can write

$$\mathbf{Q}_\omega^+(A) = \sum_{n \geq 0} \mathbf{Q}_\omega^+(A \mid \tau_\rho = n) \mathbf{Q}_\omega^+(\tau_\rho = n). \quad (6.19)$$

However if $A \in \mathcal{T}$ then $A \in \mathcal{G}_n$, for all n , hence

$$\mathbf{Q}_\omega^+(A \mid \tau_\rho = n) = \mathbf{Q}_\omega^+(\{S_{n+k}\}_{k \geq 0} \in A \mid \tau_\rho = n) = \mathbf{P}^\uparrow(\Theta^{-n}A).$$

We have already remarked that $\Theta^{-n}A \in \mathcal{T}$ for all n , hence if we show that the law \mathbf{P}^\uparrow is trivial on \mathcal{T} then from (6.19) it follows that $\mathbf{Q}_\omega^+(A) = 0$ and this completes the proof.

Let $k \in \mathbb{N}$, $\ell_1, \dots, \ell_k \in \mathbb{N}$ and set $M_n := \mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k \mid \mathcal{G}_n)$, $n > k$. Then $(M_n)_{n > k}$ is a $(\mathcal{G}_n)_{n > k}$ -inverse martingale, hence M_n converges \mathbf{P}^\uparrow -a.s. and in $L^1(d\mathbf{P}^\uparrow)$ to $M := \mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k \mid \mathcal{T})$. On the other hand, by the Markov property:

$$\begin{aligned} \mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k \mid \mathcal{G}_n) &= \mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k \mid S_n) \\ &= \left[\prod_{i=1}^k p_1^\uparrow(\ell_{i-1}, \ell_i) \right] \frac{p_{n-k}^\uparrow(\ell_k, S_n)}{p_n^\uparrow(0, S_n)}, \quad \mathbf{P}^\uparrow\text{-a.s.}, \end{aligned} \quad (6.20)$$

where $l_0 := 0$ and $p_j^\uparrow(a, b)$ is the j -th iteration of the transition kernel of \mathbf{P}^\uparrow , that is the \mathbf{P}^\uparrow -probability that S goes from a to b in j steps. Now we claim that for every $x \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \frac{p_{n-k}^\uparrow(x, S_n)}{p_n^\uparrow(0, S_n)} = 1, \quad \mathbf{P}^\uparrow\text{-a.s.} \quad (6.21)$$

Then we obtain:

$$\mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k \mid \mathcal{T}) = \left[\prod_{i=1}^k p_1^\uparrow(\ell_{i-1}, \ell_i) \right] = \mathbf{P}^\uparrow(S_i = \ell_i, i = 1, \dots, k)$$

and it follows that \mathcal{T} is independent of $\sigma(S_i : i = 1, \dots, k)$. Since this is true for all $k \in \mathbb{N}$, \mathcal{T} is independent of itself and therefore must be trivial.

It remains to prove (6.21). We recall that \mathbf{P}^\uparrow is the law of a transient Markov chain on \mathbb{R} , cf. [3], and we denote by \mathbf{P}_x^\uparrow the law with starting point $x \in \mathbb{N} \cup \{0\}$. Then we can rephrase (6.21) in the following way: for all $k, x \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \frac{\mathbf{P}_x^\uparrow(S_{n-k} = y)}{\mathbf{P}_0^\uparrow(S_n = y)} \Big|_{y=S_n} = 1, \quad \mathbf{P}^\uparrow\text{-a.s.} \quad (6.22)$$

We stress that we already know that the l.h.s. of this equation has a limit as $n \rightarrow \infty$, \mathbf{P}^\uparrow -a.s. (it suffices to give a look at the r.h.s. of (6.20) and to recall that the l.h.s. of (6.20) converge \mathbf{P}^\uparrow -a.s. by the martingale argument outlined above). Therefore it suffices

to show that, for \mathbf{P}^\dagger -a.e. $S = \{S_n\}_n$, there exists a subsequence $(n_k)_k = (n_k(S))_k$ such that the l.h.s. of (6.22) tends to 1 as $n \rightarrow \infty$ along the subsequence $(n_k)_k$.

Let us denote by $(\{R_t\}_{t \geq 0}, P)$ a standard Bessel(3) process starting from zero. Then from [4, Th. 5.1] we have that, for any fixed x , under \mathbf{P}_x^\dagger the sequence $S_n/(\sqrt{2p}\sqrt{n})$ convergence in law toward R_1 (note that $\sqrt{2p}$ is the variance of S_1 under the unperturbed random walk measure \mathbf{P}). In particular, for all $0 < a < b < \infty$ we have

$$\mathbf{P}_x^\dagger(S_n \in [a\sqrt{2p}\sqrt{n}, b\sqrt{2p}\sqrt{n}]) \rightarrow P(R_1 \in [a, b]) \quad (n \rightarrow \infty). \quad (6.23)$$

It follows that, for \mathbf{P}_x^\dagger -a.e. $S = \{S_n\}_n$, there exists a subsequence $(n_k)_k = (n_k(S))_k$ such that $\liminf_k S_{n_k}/\sqrt{n_k} > 0$ and $\limsup_k S_{n_k}/\sqrt{n_k} < \infty$: indeed, from (6.23) and by Fatou's lemma

$$\mathbf{P}_x^\dagger(S_n \in [a\sqrt{2p}\sqrt{n}, b\sqrt{2p}\sqrt{n}] \text{ i.o.}) \geq P(R_1 \in [a, b]),$$

where $\{A_n \text{ i.o.}\} := \limsup_n A_n$ for a sequence of events $(A_n)_n$, and since $P(R_1 \in [a, b])$ can be made arbitrarily close to 1 by choosing a small and b large, we obtain the claim.

Therefore, in order to prove (6.22) it is enough to show that for any sequence $(y_n)_n \subset \mathbb{N}$ such that $\liminf_n y_n/\sqrt{n} > 0$ and $\limsup_n y_n/\sqrt{n} < \infty$ we have

$$\frac{\mathbf{P}_x^\dagger(S_{n-k} = y_n)}{\mathbf{P}_0^\dagger(S_n = y_n)} \rightarrow 1 \quad (n \rightarrow \infty). \quad (6.24)$$

We denote by $p(\cdot, \cdot)$ the transition kernel of S under \mathbf{P} :

$$p(x, y) := p\mathbf{1}_{(y=x+1)} + p\mathbf{1}_{(y=x-1)} + (1-2p)\mathbf{1}_{(y=x)}, \quad x, y \in \mathbb{N},$$

and we denote by \mathbf{P}_x the law of $x + S$ under \mathbf{P} , $x \in \mathbb{N} \cup \{0\}$. We recall that the transition probability kernel $p^\dagger(x, y) = p_1^\dagger(x, y)$ of S under \mathbf{P}^\dagger is a h -transform of $p(x, y)\mathbf{1}_{(y>0)}$:

$$p^\dagger(x, y) = p(x, y)\mathbf{1}_{(y>0)} \frac{h(y)}{h(x)}, \quad x, y \geq 0,$$

where $h : \mathbb{N} \cup \{0\} \mapsto (0, \infty)$ satisfies:

$$\sum_y p(x, y)\mathbf{1}_{(y>0)} h(y) = h(x), \quad x \geq 0,$$

see [3]. It is easy to see that necessarily $h(x) = h(0)x/p$ for all $x \geq 1$, hence for all $x, y \geq 1$:

$$\begin{aligned} \frac{\mathbf{P}_x^\dagger(S_{n-k} = y)}{\mathbf{P}_0^\dagger(S_n = y)} &= \frac{\mathbf{P}_x(S_n = y, S_1 > 0, \dots, S_n > 0)}{\mathbf{P}_0(S_n = y, S_1 > 0, \dots, S_n > 0)} \cdot \frac{p}{x} \\ &= \frac{\mathbf{P}_x(S_n = y) - \mathbf{P}_x(S_n = -y)}{p[\mathbf{P}_0(S_{n-1} = y-1) - \mathbf{P}_0(S_{n-1} = y+1)]} \cdot \frac{p}{x}, \end{aligned}$$

where we have used the reflection principle. The Local Limit Theorem given by [20, Th. 13 in Ch. VII.3] yields the expansion:

$$\mathbf{P}_0(S_n = z \cdot \sqrt{n}) = \gamma(z) \left[1 + \frac{c}{\sqrt{n}}(z^3 - 3z) \right] + o(1/\sqrt{n}),$$

uniformly in $z \in n^{-1/2}\mathbb{N}$, where $\gamma(\cdot)$ is the density of $\mathcal{N}(0, 1)$ and c a positive constant. Then we obtain:

$$\frac{\mathbf{P}_x(S_n = y_n) - \mathbf{P}_x(S_n = -y_n)}{\mathbf{P}_0(S_{n-1} = y_n - 1) - \mathbf{P}_0(S_{n-1} = y_n + 1)} \rightarrow x, \quad (n \rightarrow \infty),$$

and the proof of (6.24) is complete. \square

Acknowledgments. G.G. acknowledges the support of GIP-ANR, project *POLINTBIO*.

REFERENCES

- [1] K. S. Alexander and V. Sidoravicius, *Pinning of polymers and interfaces by random potentials*, preprint (2005), arXiv.org: math.PR/0501028.
- [2] S. Asmussen, *Applied Probability and Queues*, Second Edition, Application of Mathematics **51**, Springer-Verlag, New York (2003).
- [3] J. Bertoin and R. A. Doney, *On conditioning a random walk to stay nonnegative*, Ann. Probab. **22** (1994), no. 4, 2152–2167.
- [4] A. Bryn-Jones and R. A. Doney, *A functional central limit theorem for random walks conditional to stay non-negative*, MIMS EPrint 2006.52 (2006), to appear in Proc. London Math. Soc.
- [5] E. Bolthausen and F. den Hollander, *Localization transition for a polymer near an interface*, Ann. Probab. **25** (1997), 1334–1366.
- [6] E. Bolthausen and G. Giacomin, *Periodic copolymers at selective interfaces: a large deviations approach*, Ann. Appl. Probab. **15** (2005), 963–983.
- [7] F. Caravenna, G. Giacomin and L. Zambotti, *Sharp asymptotic behavior for wetting models in $(1+1)$ -dimension*, preprint (2005), arXiv.org: math.PR/0511376.
- [8] F. Caravenna, G. Giacomin and L. Zambotti, *A renewal theory approach to periodic copolymers with adsorption*, preprint (2006), arXiv.org: math.PR/0507178.
- [9] J.-D. Deuschel, G. Giacomin and L. Zambotti, *Scaling limits of equilibrium wetting models in $(1+1)$ -dimension*, Probab. Theory Relat. Fields **132** (2005), 471–500.
- [10] W. Feller, *An introduction to probability theory and its applications*, Vol. II, Second edition, John Wiley & Sons (1971).
- [11] S. Galluccio and R. Graber, *Depinning transition of a directed polymer by a periodic potential: a d -dimensional solution*, Phys. Rev. E **53** (1996), R5584–R5587.
- [12] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter Studies in Mathematics, Berlin – New York (1988).
- [13] G. Giacomin, *Localization phenomena in random polymer models*, preprint (2004), available on the web page of the author.
- [14] G. Giacomin and F. L. Toninelli, *Estimates on path delocalization for copolymers at selective interfaces*, Probab. Theor. Rel. Fields **133** (2005), 464–482.
- [15] G. Giacomin and F. L. Toninelli, *The localized phase of disordered copolymers with adsorption*, ALEA **1** (2006), 149–180.
- [16] Y. Isozaki and N. Yoshida, *Weakly pinned random walk on the wall: pathwise descriptions of the phase transition*, Stochastic Process. Appl. **96** (2001), 261–284.
- [17] C. Monthus, T. Garel and H. Orland, *Copolymer at a selective interface and two dimensional wetting: a grand canonical approach*, Eur. Phys. J. B **17** (2000), 121–130.
- [18] A. Naidedov and S. Nechaev, *Adsorption of a random heteropolymer at a potential well revisited: location of transition point and design of sequences*, J. Phys. A: Math. Gen. **34** (2001), 5625–5634.
- [19] S. Nechaev and Y.-C. Zhang, *Exact solution of the 2D wetting problem in a periodic potential*, Phys. Rev. Lett. **74** (1995), 1815–1818.
- [20] V. V. Petrov, *Sums of independent random variables*, Springer-Verlag, New York–Heidelberg (1975).
- [21] A. Rechnitzer and E. J. Janse van Rensburg, *Exchange relations, Dyck paths and copolymer adsorption*, Discrete Appl. Math. **140** (2004), 49–71.
- [22] J.-U. Sommer and M. Daoud, *Copolymers at selective interfaces*, Europhys. Lett. **32(5)** (1995), 407–412.
- [23] J.-U. Sommer and M. Daoud, *Adsorption of multiblock copolymers at interfaces between selective solvents: Single-chain properties*, Phys. Rev. E **53** (1996), 905–920.
- [24] C. E. Soteros and S. G. Whittington, *The statistical mechanics of random copolymers*, J. Phys. A: Math. Gen. **37** (2004), R279–R325.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH
E-mail address: `francesco.caravenna@math.unizh.ch`

LABORATOIRE DE PROBABILITÉS DE P 6 & 7 (CNRS U.M.R. 7599) AND UNIVERSITÉ PARIS 7 – DENIS
DIDEROT, U.F.R. MATHÉMATIQUES, CASE 7012, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE
E-mail address: `giacomini@math.jussieu.fr`

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI 32, 20133
MILANO, ITALY
E-mail address: `lorenzo.zambotti@polimi.it`